Modeling and numerical simulation of floating structures in shallow-water flows

Sacha Cardonna[1], David Lannes[2], Fabien Marche[1] & François Vilar[1]

[1] *Institute of Mathematics Alexander Grothendieck, University of Montpellier, France*
[2] *Institute of Mathematics of Bordeaux, University of Bordeaux, France*

Congrès des Jeunes Chercheur.e.s en Mathématiques Appliquées
ENPC, Champs-sur-Marne, France – March 2026

CJC-MA 5ème édition 2026

UNIVERSITÉ DE MONTPELLIER

IMAG INSTITUT MONTPELLIERAIN ALEXANDER GROTHENDIECK

Overview of this talk

Modeling and numerical simulation of floating structures in shallow-water flows

- **Modeling and numerical simulation**: building system of equations and algorithms to approximate their solutions,
- **Floating structures**: rigid objects floating on water and interacting with waves,
- **Shallow-water flows**: simplified models describing water motion in shallow environments.

Table of contents

Slides available online at sachacardonna.github.io

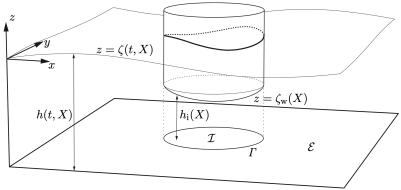
Table of contents

Slides available online at sachacardonna.github.io

Why do we do what we do?

Understanding and predicting the interaction between waves and floating structures is a key issue in many practical and scientific contexts:

- **Energy and offshore applications**: floating devices such as wave energy converters, platforms, or breakwaters interact strongly with incoming waves,
- **Safety and design**: reliable models are needed to assess loads, stability, and long-term behavior of floating structures,
- **Scientific challenges**: wave–structure interactions involve nonlinear effects and constraints that require dedicated mathematical models and numerical tools.



From the theory...

... to its potential applications

Why is this non-trivial?

Wave–structure interactions have been studied for a long time, and many models are available in the literature. But several issues remain:

- **Modeling challenges**: classical models capture important physical effects, but their mathematical structure is often complex or only partially understood,
- **Well-posedness issues**: in many situations, existence, uniqueness, or stability of solutions are not fully established,
- **Numerical challenges**: robust simulations require schemes able to handle nonlinearities, constraints, and strong wave–structure interactions.

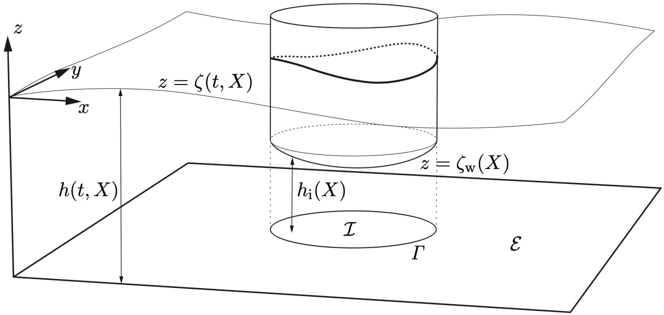
Table of contents

Slides available online at sachacardonna.github.io

Table of contents

Slides available online at sachacardonna.github.io

3D setting



- Fluid domain bounded below by a flat bottom and above by a free surface $z = \zeta(\boldsymbol{x}, t)$,
- A rigid, stationnary, partially immersed floating structure interacts with the surrounding flow,
- Vertical variations are averaged, leading to shallow-water type models.

2D horizontal setting

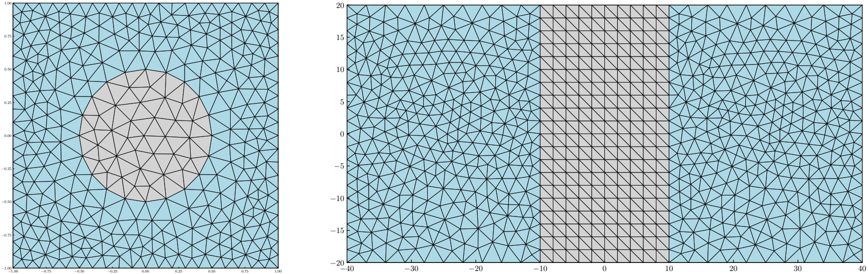


Figure: Two type of configurations: cylinder object and pontoon-like object.

For the latter numerical resolution, we will work in a 2D horizontal setting, where the domain is decomposed into three parts:

- Ω_s ("solid region"): hor. projection of the wet part under the object,
- Ω_f ("fluid region"): hor. projection of the free surface in contact with air,
- Ω_{fs}: interface separating them, with unit normal $\boldsymbol{n}$ pointing toward Ω_f.

Table of contents

Slides available online at sachacardonna.github.io

Baseline model for water waves

Nonlinear shallow-water (NSW) equations

$$\begin{cases} \partial_t \zeta + \nabla_{\boldsymbol{x}} \cdot (H\boldsymbol{u}) = 0, \\ \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u} + g\nabla_{\boldsymbol{x}} \zeta = -\frac{1}{\rho} \nabla_{\boldsymbol{x}} \underline{p}, \end{cases}$$

- $\zeta : \mathbb{R}^2 \times (0, T) \mapsto \zeta(\boldsymbol{x}, t) \in \mathbb{R}$ is the **free surface elevation** with respect to the rest state,
- $\boldsymbol{u} : \mathbb{R}^2 \times (0, T) \mapsto \boldsymbol{u}(\boldsymbol{x}, t) \in \mathbb{R}^2$ is the **depth-averaged horizontal velocity**,
- $H : \mathbb{R}^2 \times (0, T) \mapsto H(\boldsymbol{x}, t) := H_0 + \zeta(\boldsymbol{x}, t) \in \mathbb{R}_+$ is the **water depth**,
- $\underline{p} : \mathbb{R}^2 \times (0, T) \mapsto \underline{p}(\boldsymbol{x}, t) \in \mathbb{R}$ is the **pressure at the free surface**.

Properties of the model

- NSW equations are a **nonlinear** system of **hyperbolic** conservation laws,
- It is valid in the **shallow-water regime** → vertical var. are negligible,
- But it does not account for **dispersive effects** → we chose the "simplest" model to focus on the coupling and numerical issues.

Table of contents

Slides available online at sachacardonna.github.io

Constraints induced by the floating object

Exterior region Ω_f (free surface)

- Surface is in contact with air,
- Pressure is prescribed: $\underline{\mathrm{p}} = \mathrm{p}_{\mathrm{atm}}$,
- The NSW momentum equation has no source term:

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u} + \mathrm{g} \nabla_{\boldsymbol{x}} \zeta = \mathbf{0}.$$

Interior region Ω_s (under the object)

- Free surface is blocked by the object (underwater part of the structure),
- Elevation is prescribed: $\zeta^s = \zeta^{\mathrm{w}}$,
- Leads to an "incompressible" constraint: $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^s = 0$.

Coupling at the interface Ω_{fs}

- Mass conservation (normal flux continuity): $H\boldsymbol{u} \cdot \boldsymbol{n} = H^s \boldsymbol{u}^s \cdot \boldsymbol{n}$;
- Pressure transmission (energy consistency): $\Pi = \Pi^s$, where $\Pi = \rho \mathrm{g} \zeta + \frac{1}{2}\rho|\boldsymbol{u}|^2$ and $\Pi^s = \underline{\mathrm{p}}^s - \mathrm{p}_{\mathrm{atm}} + \rho \mathrm{g} \zeta + \frac{1}{2}\rho|\boldsymbol{u}|^2$,
- Velocity compatibility (no artificial vortex): $\boldsymbol{u} \cdot \boldsymbol{n}^\perp = \boldsymbol{u}^s \cdot \boldsymbol{n}^\perp$.

Summary: a coupled "partly constrained" SW system

A first wave–structure interaction model

- Exterior region Ω_f (free surface):

$$\begin{aligned} \partial_t \zeta + \nabla_{\boldsymbol{x}} \cdot (H\boldsymbol{u}) &= 0 \\ \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u} + \mathrm{g}\nabla_{\boldsymbol{x}} \zeta &= \boldsymbol{0} \end{aligned}$$

nonlinear shallow-water equations

- Interior region Ω_s (under the object):

$$\begin{aligned} \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^s &= 0 \\ \partial_t \boldsymbol{u}^s + \boldsymbol{u}^s \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u}^s &= \rho^{-1} \nabla_{\boldsymbol{x}} \underline{\mathrm{p}}^s \end{aligned}$$

incompressible Euler equations

- Interface Ω_{fs} (coupling conditions):

$$H\boldsymbol{u} \cdot \boldsymbol{n} = H^s \boldsymbol{u}^s \cdot \boldsymbol{n}$$ *mass flux continuity*

$$\Pi = \Pi^s$$ *pressure continuity*

$$\boldsymbol{u} \cdot \boldsymbol{n}^\perp = \boldsymbol{u}^s \cdot \boldsymbol{n}^\perp$$ *tangential continuity*

Irrotational initial data: a key simplification

Propagation of irrotationality

If $\nabla_{\boldsymbol{x}}^{\perp} \cdot \boldsymbol{u}(\cdot, 0) = 0$ in Ω_f and $\nabla_{\boldsymbol{x}}^{\perp} \cdot \boldsymbol{u}^s(\cdot, 0) = 0$ in Ω_s, it stays true for all $t \geq 0$
↪ Physically the flow remains smooth, without rotation or vortex generation.

Consequence: the interior becomes elliptic

- Because the flow remains **irrotational**, the interior velocity can be written as a **gradient field** *i.e.* there exists a potential ϕ^s such that

$$\boldsymbol{u}^s = \nabla_{\boldsymbol{x}} \phi^s \quad \text{in } \Omega_s.$$

- Under the object, the free surface is fixed, hence the water depth is **constant in time**: $\partial_t H^s = \partial_t (H_0 + \zeta^{\mathrm{w}}) = 0$.
- Combining irrotationality with the incompressibility constraint $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^s = 0$ leads to an **elliptic problem** for the potential:

$$\nabla_{\boldsymbol{x}} \cdot \left(H^s \nabla_{\boldsymbol{x}} \phi^s \right) = 0 \quad \text{in } \Omega_s.$$

From the interior constraint to a mixed formulation

What happens under the object

- To solve $\nabla_{\boldsymbol{x}} \cdot \left(H^s \nabla_{\boldsymbol{x}} \phi^s\right) = 0$ in Ω_s, we prescribe the trace of the potential on the interface,

$$\phi^s = \psi^s \quad \text{on } \Omega_{fs}.$$

- The trace ψ^s is **not arbitrary**: it evolves in time according to an ODE obtained from the Bernoulli relation on the interface,

$$\partial_t \psi^s = -g\zeta - \frac{1}{2}|\boldsymbol{u}|^2 \quad \text{on } \Omega_{fs}.$$

How the interior talks to the exterior

We define the following **Dirichlet–Neumann operator:**

$$\Lambda\psi^s := H^s \nabla_{\boldsymbol{x}} \phi^s \cdot \boldsymbol{n} \quad \text{on } \Omega_{fs},$$

maps the potential ψ^s (Dirichlet data) to the normal flux (Neumann data)
↪ Quantifies how the interior motion exchanges water with the exterior flow

Equivalent mixed formulation (irrotational case)

Exterior hyperbolic problem with boundary coupling

- Exterior region Ω_f (free surface):

$$\begin{aligned} \partial_t \zeta + \nabla_{\boldsymbol{x}} \cdot (H\boldsymbol{u}) &= 0 \\ \partial_t \boldsymbol{u} + \nabla_{\boldsymbol{x}} \big(\mathrm{g}\zeta + \tfrac{1}{2}|\boldsymbol{u}|^2 \big) &= \boldsymbol{0} \end{aligned}$$

nonlinear shallow-water equations

- Interior region Ω_s (object):

$$\begin{aligned} \nabla_{\boldsymbol{x}} \cdot \left(H^s \nabla_{\boldsymbol{x}} \phi^s \right) &= 0 \quad \text{in } \Omega_s \\ \phi^s &= \psi^s \quad \text{on } \Omega_{fs} \end{aligned}$$

elliptic equation on potential

- Interface Ω_{fs} (boundary coupling):

$$\boldsymbol{n} \cdot (H\boldsymbol{u}) = \Lambda \psi^s$$

normal flux from the interior

$$\partial_t \psi^s = -\mathrm{g}\zeta - \tfrac{1}{2}|\boldsymbol{u}|^2$$

Bernoulli ODE

- Initial conditions: $(\zeta, \boldsymbol{u})_{|t=0} = (\zeta^{\mathrm{in}}, \boldsymbol{u}^{\mathrm{in}})$ in Ω_f and $\psi^s_{|t=0} = \psi^s_{\mathrm{in}}$ on Ω_{fs}.

Table of contents

Slides available online at sachacardonna.github.io

A special configuration: the infinite pontoon

- The interior region is an infinite strip $\Omega_s = (-\ell, \ell) \times \mathbb{R}$, while the exterior domain consists of two half-planes $\Omega_{f\pm} = \{(x, y) \mid \pm x > \ell\}$.
- In this configuration, the **Dirichlet–Neumann operator is explicit**: no elliptic problem has to be solved inside the structure.
- This makes the infinite pontoon an **ideal benchmark**: it allows us to validate the numerical coupling independently of the interior solver.
- Moreover, for y-independent solutions, the model reduces to a known **one-dimensional wave–structure interaction system**.

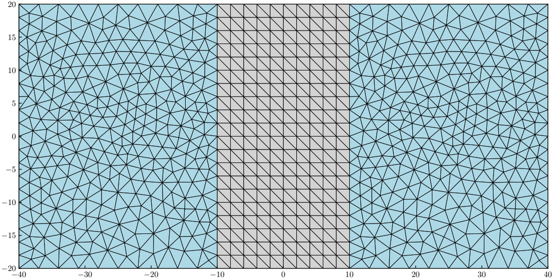


Table of contents

Slides available online at sachacardonna.github.io

Numerical challenges induced by the coupled model

What makes wave–structure simulations difficult?

- **Two types of PDEs: hyperbolic** NSW in Ω_f, **elliptic** PDE to compute DN operator on Ω_{fs} with a strong coupling,
- **Nonlinearities and discontinuities:** hyperbolicity and nonlinearity imply solutions can become **discontinuous** in finite time,
- **Geometry:** dealing with **unstructured meshes** and **complex geometries** is more challenging than simple Cartesian meshes.

What we want from the discretization

- **High-order accuracy** to capture every singularities and nonlinear effects,
- **Robustness** near shocks and wet/dry fronts (if topography is included),
- **Preservation of physical properties** (e.g. positivity of water depth, energy consistency)...

$\hookrightarrow$ These requirements are often in tension, and designing a scheme that satisfies all of them is non-trivial!

Table of contents

Slides available online at sachacardonna.github.io

Two classical approaches for hyperbolic PDEs

Finite Volume (FV)

- Integral formulation over control volumes $\omega_c \subset \Omega$ with $\Omega = \bigcup \omega_c$;
- Piecewise constant approximation:

$$\boldsymbol{v}_h^c(t) \simeq \frac{1}{|\omega_c|} \int_{\omega_c} \boldsymbol{v}(\boldsymbol{x}, t)\, d\boldsymbol{x},$$

  where $\boldsymbol{v}$ is the exact solution;
- Numerical flux $\mathbb{F}^*$ ensures conservation and stability.

✔ Robust and easy to implement, well-suited for nonlinear problems;

✘ Low-order accuracy unless polynomial reconstruction is applied.

Discontinuous Galerkin (DG)

- Weak formulation on each element $\omega_c \subset \Omega$ with $\Omega = \bigcup \omega_c$;
- Piecewise polynomial approx.:

$$\boldsymbol{v}_h^c(\boldsymbol{x}, t) = \sum_{m=1}^{\dim \mathbb{P}^k} \boldsymbol{v}_m^c(t)\, \psi_m^c(\boldsymbol{x}),$$

  with test functions in $\mathbb{P}^k(\omega_c)$;
- Numerical flux $\mathbb{F}^*$ to ensure local conservation.

✔ High-order accuracy with compact stencil, well-suited for parallelism;

✘ Less robust, more complex implementation and prone to oscillations.

Monolithic DG/FV idea (hyperbolic solver)

Best of both worlds between precision and robustness

- DG is high-order accurate but may oscillate near discontinuities,
- FV is robust (in particular positivity-friendly) but low-order,

Monolithic DG/FV schemes blend the two approaches in a single framework, with a local correction of the DG solution where needed.

Key ingredient: subcell viewpoint

Considering $\boldsymbol{v}_h^c$ the DG numerical solution on each cell $\omega_c \subset \Omega_f$, we build a sub-partition $\{S_m^c\}_{m=1}^{N_s} := \{S_1^c, S_2^c, \ldots, S_{N_s}^c\}$ and track subcell averages

$$\overline{\boldsymbol{v}}_m^c(t) = \frac{1}{|S_m^c|} \int_{S_m^c} \boldsymbol{v}_h^c(\boldsymbol{x}, t)\, d\boldsymbol{x},$$

↪ DG solution can then be rewritten as a **FV-like mean values** on subcells (where fluxes are now defined across subfaces).

A classical mesh ...

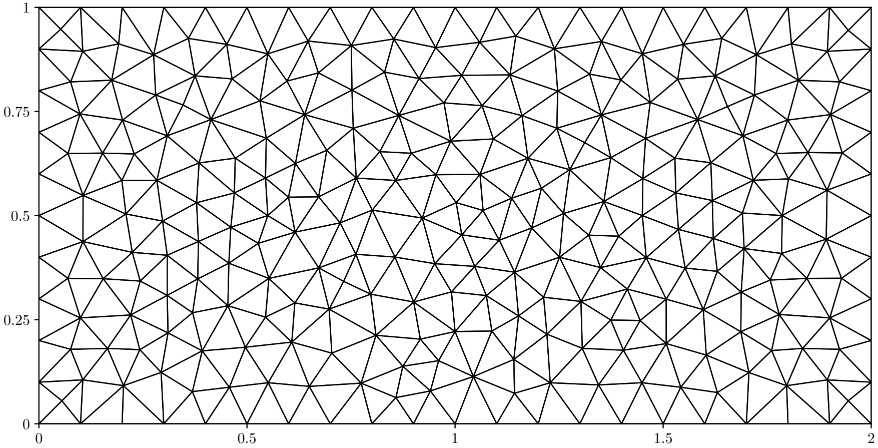


Figure: Unstructured simplicial mesh with $n_{\text{el}} = 350$ cells.

… and its subdivision

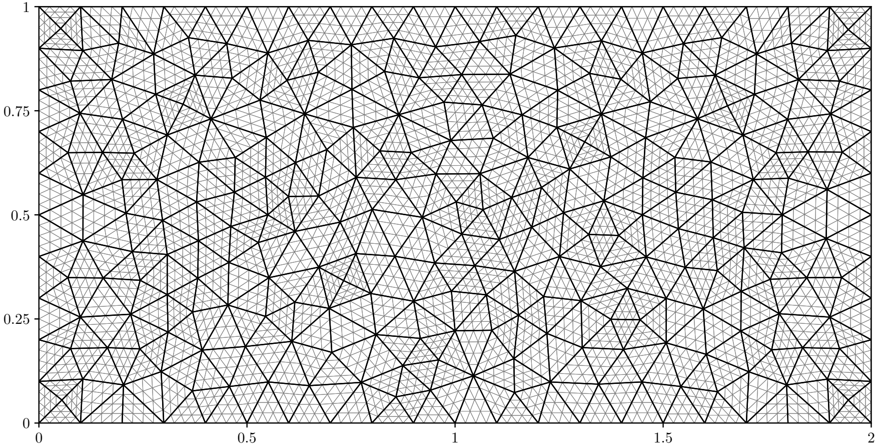


Figure: Unstructured simplicial mesh $\mathbb{P}^3$ subdivision onto triangles with $n_{\mathrm{el}} = 350$ cells.

Flux blending and fully-discrete scheme

Blended numerical flux

On each subface Γ^c_{mp} of each subcell S^c_m, we combine:

- a high-order **DG flux** $\widehat{\mathbb{F}}_{mp}$, precise but potentially oscillatory,
- a first-order **FV flux** $\mathbb{F}^{*,\mathsf{FV}}_{mp}$, very robust but less accurate,

through a convex blend

$$\widetilde{\mathbb{F}}_{mp} = \mathbb{F}^{*,\mathsf{FV}}_{mp} + \Theta_{mp}\left(\widehat{\mathbb{F}}_{mp} - \mathbb{F}^{*,\mathsf{FV}}_{mp}\right), \qquad \Theta_{mp} \in [0,1],$$

where Θ_{mp} is a blending coefficient that controls the balance between **accuracy** and **robustness**!

Local subcell monolithic DG/FV scheme for the hyperbolic part (NSW)

$$\overline{\boldsymbol{v}}^{c,n+1}_m = \overline{\boldsymbol{v}}^{c,n}_m - \frac{\Delta t^n}{|S^c_m|} \sum_{S^v_p \in Neigh(S^c_m)} |\Gamma^c_{mp}|\, \widetilde{\mathbb{F}}_{mp}, \quad \forall S^c_m \in \{S^c_j\}_{j=1}^{N_s}, \quad \forall \omega_c \subset \Omega_f.$$

Table of contents

Slides available online at sachacardonna.github.io

Hybrid High-Order (HHO) for the interior elliptic problem

Why HHO in Ω_s?

Hybrid High-Order (HHO) methods are a modern class of numerical schemes designed for **elliptic** and **parabolic** PDEs, offering high-order accuracy and flexibility on general meshes → Ideal solver for the elliptic problem in Ω_s!

Discrete unknowns

Cell and face unknowns (polynomials of degree k):

$$\phi_c \in \mathbb{P}^k(\omega_c), \qquad \phi_\Gamma \in \mathbb{P}^k(\Gamma), \quad \omega_c \subset \Omega_s, \ \Gamma \subset \partial\omega_c.$$

- **Hybrid viewpoint:** cell unknowns represent the interior behaviour, while face unknowns control how neighbouring cells communicate,
- **Local reconstruction:** a high-order approximation of the gradient is built inside each cell, reproducing the continuous integration-by-parts structure,
- **Computational efficiency:** cell unknowns are eliminated locally, so the global problem involves face unknowns only.

Table of contents

Slides available online at sachacardonna.github.io

Coupling everything, everywhere, all at once

Why SSP–Runge–Kutta?

High-order explicit time integrators built as **convex combinations of Euler steps**, therefore **inheriting Euler stability properties**!

One time step: what the algorithm actually does

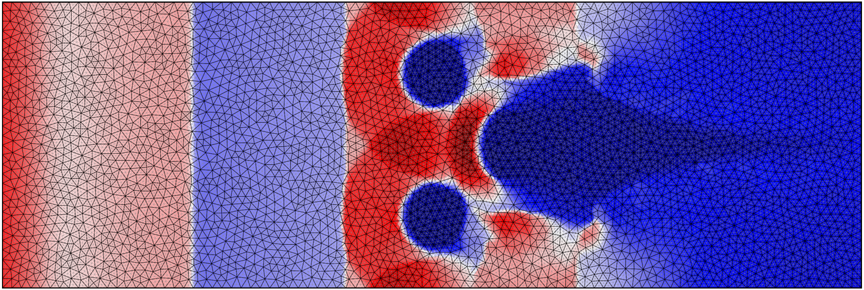
At each time step, we simply alternate the following operations:

1. **Read the fluid state** near the obstacle (water height and velocity),
2. **Advance the interface potentials** using the Bernoulli ODE,
3. **Compute the normal discharge** through the object by solving the interior elliptic problem,
4. **Feed this discharge** as boundary conditions to the fluid solver,
5. **Advance the fluid** with the DG/FV subcell scheme.

Table of contents

Slides available online at sachacardonna.github.io

Finally, some waves



1. Wave propagation under a fixed floating obstacle (MP4),
2. Wave hitting a submerged moving obstacle (MP4),
3. Wave generation by a prescribed motion of the floating structure (MP4).

↪ More simulations on my webpage!

Lies?

Bro you said the object was fixed?

Yes, I did... However, in the pseudo-1D configuration, the model/numerics can actually be extended to:

- **Prescribed motion:** impose a given vertical trajectory to the object (this is the case in the wave-maker simulation),
- **Free motion:** couple the fluid system with an additional ODE expressing Newton's law for the vertical dynamics of the structure.

But I think we already have enough equations for today ☺

1. Wave propagation under a fixed floating obstacle (MP4),
2. Wave hitting a submerged moving obstacle (MP4),
3. Wave generation by a prescribed motion of the floating structure (MP4).

↪ More simulations on my webpage!

Table of contents

Slides available online at sachacardonna.github.io

Ongoing and upcoming work

What has been done...

S.C., A. Haidar, F. Marche & F. Vilar, *Local subcell monolithic DG/FV methods for nonlinear SW models with source terms*. Submitted. 2025.

S.C., F. Marche & F. Vilar, *An high-order scheme for 2D NSW equations with topography and friction effects on unstructured grids*. Submitted. 2026.

S.C., D. Lannes, F. Marche & F. Vilar, *Numerical resolution of 2D NSW equations with a partly immersed surface obstacle*. In preparation. 2026.

... and what are the plans for the future!

- Designing a model taking into account the **free motion** of the structure,
- Adaptation of the method to **moving** or **deforming** meshes via an **ALE framework**,
- Extension to **dispersive water-waves equations** (e.g. Green–Naghdi, Boussinesq) to capture more complex wave phenomena.

∼ Thank you for your attention! ∼

Special acknowledgments

- **Organizing committee of CJC-MA26**

  ↪ *M. Chassard, G. Chennetier, S. Delannoy-Pavy, P. Lissy, A. Massimini, D. Mallitasig & J. Najim*

- **My mentors and advisors**

  ↪ *F. Marche & F. Vilar*

- **Supportive professors**

  ↪ *A. Duran, D. Lannes, B. Mohammadi, P. Azerad & D. Le Roux*

- **Colleagues and friends**

  ↪ *A. Haidar & M. Hanot*

Contact: *sacha.cardonna@umontpellier.fr*
Website: *sachacardonna.github.io*