Nonlinear Shallow Water equations

The study of Shallow Water equations holds significant importance in the field of applied mathematics and fluid dynamics due to its capacity to model hydrodynamic flows in environments where the horizontal length scales are much larger than the vertical depth, such as oceans, rivers, and coastal areas.

The 1D pre-balanced Nonlinear Shallow Water equations can be described as an hyperbolic system with source term:

\[ \partial_t \psi + \partial_x F(\psi, b) = B(\psi, \partial_x b) \]

where the physical functions are defined as follows, with \( H = \{(q, \eta) \in \mathbb{R}^2 : H := \eta - q \geq 0 \} \):

- \( b: \mathbb{R} \rightarrow \mathbb{R} \) is the topography parametrization;
- \( \psi: \mathbb{R} \times \mathbb{R}_+ \rightarrow H \) is the vector gathering total elevation \( \psi \) and discharge \( \eta \);
- \( F: H \times H \rightarrow \mathbb{R}^d \) is the nonlinear flux function;
- \( B: H \times H \rightarrow \mathbb{R}^2 \) is the source term depending on topography.

These equations, obtained by theoretical derivation of Euler system, provide a simplified yet powerful framework for understanding and predicting phenomena such as tidal waves, tsunamis, and storm surges, offering crucial insights for disaster preparedness and environmental management. Furthermore, they serve as a fundamental tool in the numerical analysis for the development of efficient computational algorithms, enhancing the accuracy and reliability of simulations in geo-flows.

Issues of DG schemes on nonlinear hyperbolic problems

- The Discontinuous Galerkin (DG) finite element method is interesting for its ability to combine features from both Finite Volume (FV) and Finite Element (FE) methods, offering high accuracy and flexibility in handling complex geometries and adaptive mesh refinements. Moreover, its local discontinuous formulation makes it highly parallelizable and suitable for large-scale computations on modern high-performance computing architectures, enhancing its applicability to a wide range of scientific problems.

- Nevertheless, when dealing with nonlinear hyperbolic equations, we can observe the presence of non-physical oscillations when approaching discontinuities or strong gradients, leading to, for example, non-preservation of maximum principle, or/and loss of water height possibility in Shallow Water context, even for smooth solutions.

Assembling DG accuracy with FV robustness

We introduce a novel discretization method termed Monolithic DG-FV subcell convex property preserving schemes. This method is grounded in a Discontinuous Galerkin (DG) formalism with an arbitrary order of accuracy and interprets the DG scheme as a Finite-Volume (FV)-like approach on a sub-partition, characterized by reconstructed high-order interface fluxes.

Subcell decomposition of a cell \( \omega \) through \( k + 2 \) sub-flux points.

The method lies on the incorporation of blended fluxes, which are convex combinations of reconstructed DG fluxes with first-order FV fluxes. The blending coefficients are meticulously computed for each time-step and each subcell interface to ensure adherence to several critical convex properties, including the discrete maximum principle, water-height positivity, and entropy preservation.

Monolithic subcell convex property preserving scheme

\textbf{1st step - Reformulation of DG scheme to subcell scale}

Theorem. Let \( H = \bigcup_{i=1}^n \mathbb{R}^2 \). The NSW-DG global formulation

\[ \sum_{i=1}^n (\partial_t \psi_i + \partial_x F_i(\psi_i, b_i)) = \sum_{i=1}^n B_i \psi_i, \]

for all \( \psi \in \mathbb{P}^d (\mathbb{B}) \), can be written as the subcell FV-like following semi-discrete formulation:

\[ \partial_t \psi_m = - \frac{1}{S_m} \left[ F_m \psi_m + B_m \right], \quad \forall m \in \{1, \ldots, n\}, \]

where \( [\psi_m] = ([\psi_1], [\psi_n])^T \) and \( [B_m] \) are respectively the mean values of \( \psi \) and \( B \) on the subcell \( S_m \), i.e.

\[ \psi_m := \frac{1}{S_m} \int_{S_m} \psi \, dx, \quad B_m := \frac{1}{S_m} \int_{S_m} B \, dx. \]

\textbf{2nd step - Numerical scheme}

Considering a mesh element \( \omega = (x_{j-1/2}, x_{j+1/2}) \) such that \( \mathbb{B} \subset \bigcup_{i=1}^n \mathbb{B}^i \), the \( n \) interior blended fluxes expression are given by

\[ F_m \psi_m = \frac{\psi_{m+1}}{F_{m+1}} - \frac{\psi_m}{F_m} + \frac{\psi_{m-1}}{F_{m-1}} - \frac{\psi_m}{F_m} = \frac{\psi_{m+1}}{F_{m+1}} - \frac{\psi_{m-1}}{F_{m-1}} + \frac{\psi_m}{F_m} - \frac{\psi_m}{F_m}, \]

where we take the blending coefficient \( \theta_m := \min \left\{ \xi_{\omega_{m+1}}, \xi_{\omega_{m-1}} \right\} \). The first-order FV flux \( \varphi_{\omega_m} \) and \( \psi \) the DG high-order reconstructed flux.

\textbf{Proposition. Considering a Forward Euler time integration, the monolithic DG-FV convex property preserving scheme is}

\[ \psi_{m+1} = \psi_m - \Delta t \left( \frac{\psi_{m+1}}{F_{m+1}} - \frac{\psi_m}{F_m} + \frac{\psi_{m-1}}{F_{m-1}} - \frac{\psi_m}{F_m} \right). \]

\textbf{3rd step - Computation of blending coefficient}

Theoretical proofs on blending coefficient can get us explicit formulas to:

- Preserve at subcell scale a local maximum principle with \( \theta^{\omega_{m+1}, \omega_{m-1}} \):
- Ensure that our solution is in the open convex set \( H^* \), i.e. preserving water height positivity with \( \theta^{\omega_{m+1}, \omega_{m-1}} \):
- Have conservation/dissipation of entropies on cell/subcell scale with \( \theta^B \).

We just need to compute every blending coefficient for each time step at each subcell interface, and taking the minimum of desired properties to assure the stability of our method:

\[ \theta_{m+1} := \min \left\{ \xi_{\omega_{m+1}}, \theta^{\omega_{m+1}, \omega_{m-1}} \psi, \theta^B \right\} \subset [0, 1]. \]

Numerical validations

\[ \begin{array}{c}
\text{Dam-break on wet bed (2D solution on 50 cells)} \\
\text{Order computation for 2D (k = 1, 2, 3)} \\
\text{Dam-break on dry bed (2D solution on 10 cells)}
\end{array} \]

Ongoing & Upcoming work

1. Adapting this stabilization method on two-dimensional Shallow Water equations, in order to compute realistic simulations;
2. Searching for a theoretical model combining 2D SW equations with a floating object;
3. Developing a new scheme solving the new model with this family of numerical methods.

Contribution statement

- F. Marche1 & F. Vilar2 conceived this work inspired from F. Vilar schemes;
- S. Cardonna3 & A. Haidar2 developed the numerical formalism under the supervision of F. Vilar. They performed the analytic calculations and the numerical simulations, via a C++ code, originally provided by F. Marche and widely developed by them.

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