Monolithic DG-FV subcell convex property preserving scheme for Shallow Water

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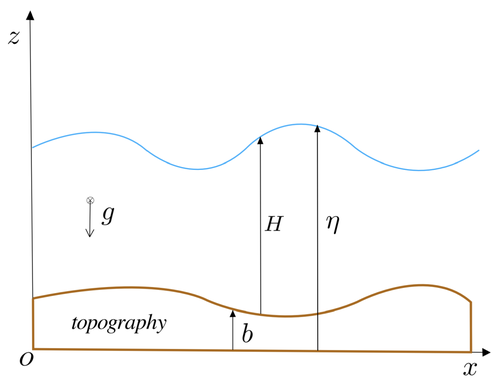
Nonlinear Shallow Water equations

The study of Shallow Water equations holds significant importance in the field of applied mathematics and fluid dynamics due to its capacity to model hydrodynamic flows in environments where the horizontal length scales are much larger than the vertical depth, such as oceans, rivers, and coastal areas.
The 1D *pre-balanced* Nonlinear Shallow Water equations can be described as an hyperbolic system with source term:

$$\partial_t \boldsymbol{v} + \partial_x \boldsymbol{F}(\boldsymbol{v}, b) = \boldsymbol{B}(\boldsymbol{v}, \partial_x b) \quad \Longleftrightarrow \quad \begin{cases} \partial_t \eta + \partial_x q = 0 \\ \partial_t q + \partial_x \left(uq + \frac{g(\eta^2 - 2\eta b)}{2} \right) = -g\eta \partial_x b \end{cases}$$

where the physical functions are defined as follows, with $\mathcal{H}^+ = \{(\eta, q) \in \mathbb{R}^2 \mid H := \eta - b \geq 0\}$:

- $b : \mathbb{R} \to \mathbb{R}$ is the *topography* parametrization;
- $\boldsymbol{v} : \mathbb{R} \times \mathbb{R}_+ \to \mathcal{H}^+$ is the vector gathering *total elevation* η and *discharge* q;
- $\boldsymbol{F} : \mathcal{H}^+ \times \mathbb{R} \to \mathbb{R}^2$ is the nonlinear flux function;
- $\boldsymbol{B} : \mathcal{H}^+ \times \mathbb{R} \to \mathbb{R}^2$ is the source term depending on topography.



These equations, obtained by theoretical derivation of Euler system, provide a simplified yet powerful framework for understanding and predicting phenomena such as tidal waves, tsunamis, and storm surges, offering crucial insights for disaster preparedness and environmental management. Furthermore, they serve as a fundamental tool in the numerical analysis for the development of efficient computational algorithms, enhancing the accuracy and reliability of simulations in geophysical flows.

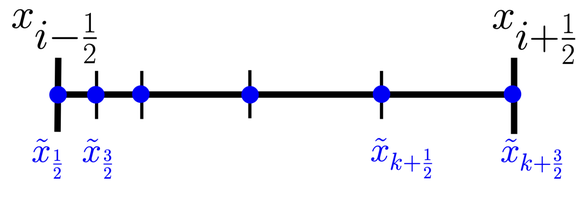
Issues of DG schemes on nonlinear hyperbolic problems

☺ The **Discontinuous Galerkin** (DG) finite element method is interesting for its ability to combine features from both *Finite Volume* (FV) and *Finite Element* (FE) methods, offering **high accuracy** and **flexibility** in handling complex geometries and adaptive mesh refinements. Moreover, its local discontinuous formulation makes it highly **parallelizable** and suitable for **large-scale computations** on modern high-performance computing architectures, enhancing its applicability to a wide range of scientific problems.

☹ Nevertheless, when dealing with nonlinear hyperbolic equations, we can observe the presence of **non-physical oscillations** when approaching discontinuities or strong gradients, leading to, for example, **non-preservation of maximum principle**, or/and **loss of water height positivity** in Shallow Water context, even for smooth solutions.

Assembling DG accuracy with FV robustness

We introduce a novel discretization method termed **Monolithic DG-FV subcell convex property preserving schemes**. This method is grounded in a Discontinuous Galerkin (DG) formalism with an arbitrary order of accuracy and interprets the DG scheme as a Finite-Volume (FV)-like approach on a sub-partition, characterized by reconstructed high-order interface fluxes.



Subcell decomposition of a cell ω_i through $k+2$ sub-flux points.

The method lies on the incorporation of **blended fluxes**, which are **convex combinations** of reconstructed DG fluxes with first-order FV fluxes. The blending coefficients are meticulously computed for each time step and each subcell interface to **ensure adherence to several critical convex properties**, including the discrete maximum principle, water-height positivity, and entropy preservation.

Monolithic subcell convex property preserving scheme

1^{st} step $-$ Reformulation of DG scheme to subcell scale

Theorem. *Let $\Omega \subset \mathbb{R}$ such that $\overline{\Omega} = \bigcup_{\omega \in \mathscr{T}_h} \overline{\omega}$. The NSW-DG global formulation*

$$\sum_{\omega \in \mathscr{T}_h} \int_\omega \partial_t \boldsymbol{v}_h \varphi \mathrm{d}x - \sum_{\omega \in \mathscr{T}_h} \int_\omega \boldsymbol{F}_h \partial_x \varphi \mathrm{d}x + \sum_{\omega \in \mathscr{T}_h} \left[\varphi \boldsymbol{\mathcal{F}}\right]_{\partial\omega} = \sum_{\omega \in \mathscr{T}_h} \int_\omega \boldsymbol{B}_h \varphi \mathrm{d}x,$$

for all $\varphi \in \mathbb{P}^k(\mathscr{T}_h)$, can be written as the subcell FV-like following semi-discrete formulation:

$$\partial_t \overline{\boldsymbol{v}}_\omega = -\frac{1}{|S_m^\omega|} \left(\widehat{\boldsymbol{F}}^\omega_{m+\frac{1}{2}} - \widehat{\boldsymbol{F}}^\omega_{m-\frac{1}{2}} \right) + \overline{\boldsymbol{B}}^\omega_m, \qquad \forall m \in [\![1, k+1]\!],$$

where $\overline{\boldsymbol{v}}^\omega_m = (\overline{\eta}^\omega_m, \overline{q}^\omega_m)^t$ and $\overline{\boldsymbol{B}}^\omega_m$ are respectively the mean values of $\boldsymbol{v}^\omega$ and $\boldsymbol{B}^\omega$ on the subcell S^ω_m, i.e.

$$\overline{\boldsymbol{v}}^\omega_m := \frac{1}{|S^\omega_m|} \int_{S^\omega_m} \boldsymbol{v}^\omega_h \mathrm{d}x, \qquad \overline{\boldsymbol{B}}^\omega_m := \frac{1}{|S^\omega_m|} \int_{S^\omega_m} \boldsymbol{B}^\omega_h \mathrm{d}x.$$

2^{nd} step $-$ Numerical scheme

Considering a mesh element $\omega_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ such that $\overline{\omega_i} := \bigcup_{m \in [\![1,k+1]\!]} \overline{S}^i_m$, the m interior blended fluxes expression are given by

$$\widetilde{\boldsymbol{F}}_{m+\frac{1}{2}} = \boldsymbol{\mathcal{F}}^{*,\mathsf{FV}}_{m+\frac{1}{2}} + \Theta_{m+\frac{1}{2}} \left(\widehat{\boldsymbol{F}}_{m+\frac{1}{2}} - \boldsymbol{\mathcal{F}}^{*,\mathsf{FV}}_{m+\frac{1}{2}} \right),$$

where we take the blending coefficient $\Theta_{m+\frac{1}{2}} := \mathrm{diag}\left(\theta_{m+\frac{1}{2}}, \theta_{m+\frac{1}{2}}\right)$, the first-order FV flux $\boldsymbol{\mathcal{F}}^{*,\mathsf{FV}}$, and $\widehat{\boldsymbol{F}}$ the DG high-order reconstructed flux.

Proposition. *Considering a Forward Euler time integration, the monolithic DG-FV convex property preserving scheme is*

$$\overline{\boldsymbol{v}}^{i,n+1}_m = \overline{\boldsymbol{v}}^{i,n}_m - \frac{\Delta t^n}{|S^i_m|} \left(\widetilde{\boldsymbol{F}}^i_{m+\frac{1}{2}} - \widetilde{\boldsymbol{F}}^i_{m-\frac{1}{2}} \right) + \Delta t^n \overline{\boldsymbol{B}}^i_m.$$

3^{rd} step $-$ Computation of blending coefficient

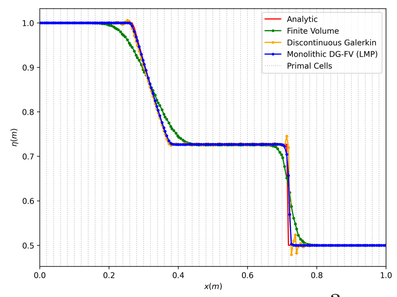
Theoretical proofs on blending coefficient can get us explicit formulas to:

- Preserve at subcell scale a *local maximum principle* with θ^{LMP};
- Ensure that our solution is in the open convex set $\mathcal{H}^+$, i.e. preserving water height positivity with $\theta^{\mathcal{H}^+}$;
- Have conservation/dissipation of entropies on cell/subcell scale with θ^{S}.

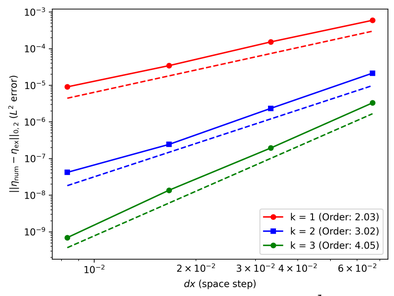
We just need to compute every blending coefficient for each time step at each subcell interface, and taking the minimum of desired properties to assure the stability of our method:

$$\theta_{m+\frac{1}{2}} := \min\left(\theta^{\mathsf{LMP}}_{m+\frac{1}{2}}, \theta^{\mathcal{H}^+}_{m+\frac{1}{2}}, \theta^{\boldsymbol{S}}_{m+\frac{1}{2}} \right) \in [0, 1].$$

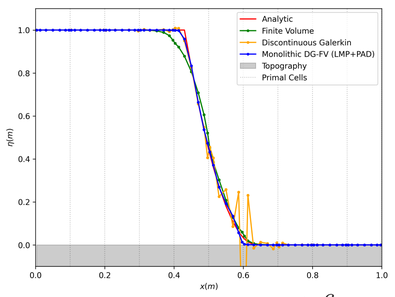
Numerical validations



Dam-break on wet bed ($\mathbb{P}^3$ solution on 50 cells)



Order computation for $\mathbb{P}^k$ $(k = 1, 2, 3)$



Dam-break on dry bed ($\mathbb{P}^6$ solution on 10 cells)

Dam-break on wet bed. Evaluates the scheme's ability to accurately simulate the dynamics of rapid water flow over an initially wet surface, crucial for predicting flood wave propagation and interaction with existing water bodies.

Dam-break on dry bed. Assesses how well the scheme transitions from dry to wet conditions, vital for flood risk analysis in dry areas. Classical DG schemes' limitations in handling wet-dry interfaces often result in numerical issues.

Order computation. Essential for assessing a numerical scheme's accuracy and efficiency through error reduction relative to mesh refinement, guiding scheme selection or enhancement for complex fluid dynamic (computation realized on a smooth solution).

ONGOING & UPCOMING WORK

1. Adapting this stabilization method on two-dimensional Shallow Water equations, in order to compute realistic simulations;
2. Searching for a theoretical model combining 2D SW equations with a floating object;
3. Developing a new scheme solving the new model with this family of numerical methods.

CONTRIBUTION STATEMENT

- F. Marche[1] & F. Vilar[1] conceived this work inspiring from F. Vilar schemes;
- S. Cardonna[1] & A. Haidar[2] developped the numerical formalism under the supervision of F. Vilar. They performed the analytic calculations and the numerical simulations, via a C++ code, originally provided by F. Marche and widely developed by them.

MORE INFORMATION

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