

High-order numerical methods for shallow-water flows on unstructured meshes: moving boundaries and wave-structure interactions

Sacha Cardonna

Supervised by Fabien Marche & François Vilar

Thesis defense for the degree of Philosophiæ Doctor
University of Montpellier, France — October 2nd, 2026



UNIVERSITÉ DE
MONTPELLIER

High-order numerical methods for shallow-water flows on unstructured meshes: moving boundaries and wave-structure interactions

High-order numerical methods: accurate and robust discretizations for complex time-dependent PDEs,

Shallow-water flows: depth-averaged models describing free-surface wave propagation,

Unstructured meshes: flexible spatial discretizations adapted to complex geometries,

Moving boundaries: Arbitrary Lagrangian–Eulerian techniques for moving computational domains,

Wave–structure interactions: coupled dynamics between shallow-water waves and floating rigid bodies.

Table of contents

1. **Introduction**
2. **A new high-order framework for shallow-water asymptotics**
 - Nonlinear shallow-water equations
 - Classical numerical methods for hyperbolic systems
 - High-order robust subcell monolithic DG/FV schemes
 - Arbitrary Lagrangian–Eulerian formulation
3. **Waves interacting with floating structures in shallow-water regimes**
 - Physical setting and constraints
 - Governing equations for water waves
 - Coupling with a floating object
 - Numerical scheme
4. **Conclusion and perspectives**



Slides available online at sachacardonna.github.io

Table of contents

1. Introduction

2. A new high-order framework for shallow-water asymptotics

Nonlinear shallow-water equations

Classical numerical methods for hyperbolic systems

High-order robust subcell monolithic DG/FV schemes

Arbitrary Lagrangian–Eulerian formulation

3. Waves interacting with floating structures in shallow-water regimes

Physical setting and constraints

Governing equations for water waves

Coupling with a floating object

Numerical scheme

4. Conclusion and perspectives



Slides available online at sachacardonna.github.io

Why are we interested in wave–structure interactions?

Wave–structure interactions arise in many applications where waves interact with floating or partially immersed structures.

- ▶ **Offshore engineering**: wave energy converters, floating platforms, coastal protection devices,
- ▶ **Nonlinear wave dynamics**: understanding how waves propagate under geometric constraints and transmission conditions,
- ▶ **Predictive simulation tools**: deriving models and numerical methods able to describe complex configurations while remaining efficient.



Figure: Wave energy converters: Triton by Oscilla Power (left) and Eco Wave Power array (right).

Why is this non-trivial?

Even in simplified shallow-water regimes, wave–structure interactions lead to coupled systems with several mathematical and numerical difficulties.

- ▶ **Strong nonlinear coupling:** the fluid and the structure influence each other through pressure, fluxes, and interface conditions,
- ▶ **Geometric constraints:** immersed or floating structures introduce constrained regions, moving interfaces, and transmission conditions,
- ▶ **Analytical issues:** the well-posedness and stability of the resulting formulations may strongly depend on the chosen coupling strategy,
- ▶ **Numerical issues:** robust schemes must preserve the relevant structure of the model while handling nonlinear waves, source terms, and constraints.

💡 *In the following, we will focus on **mathematically well-posed models** rather than purely **engineering-oriented** ones, as we are concerned mainly with the mathematical and numerical challenges of the coupling.*

Table of contents

1. Introduction

2. A new high-order framework for shallow-water asymptotics

- Nonlinear shallow-water equations

- Classical numerical methods for hyperbolic systems

- High-order robust subcell monolithic DG/FV schemes

- Arbitrary Lagrangian–Eulerian formulation

3. Waves interacting with floating structures in shallow-water regimes

- Physical setting and constraints

- Governing equations for water waves

- Coupling with a floating object

- Numerical scheme

4. Conclusion and perspectives



Slides available online at sachacardonna.github.io

Table of contents

1. Introduction
2. **A new high-order framework for shallow-water asymptotics**
 - Nonlinear shallow-water equations
 - Classical numerical methods for hyperbolic systems
 - High-order robust subcell monolithic DG/FV schemes
 - Arbitrary Lagrangian–Eulerian formulation
3. Waves interacting with floating structures in shallow-water regimes
 - Physical setting and constraints
 - Governing equations for water waves
 - Coupling with a floating object
 - Numerical scheme
4. Conclusion and perspectives



Slides available online at sachacardonna.github.io

A first-order nonlinear hyperbolic system for shallow-water flows

Nonlinear shallow-water (NSW) equations

Let $\Omega \subset \mathbb{R}^2$ be an open domain. We consider the following system:

$$\begin{cases} \partial_t \eta + \nabla_{\mathbf{x}} \cdot \mathbf{q} = 0, \\ \partial_t \mathbf{q} + \nabla_{\mathbf{x}} \cdot \left(\mathbf{u} \otimes \mathbf{q} + \frac{g\eta}{2}(\eta - 2b)\mathbb{I}_2 \right) = -g\eta \nabla_{\mathbf{x}} b. \end{cases}$$

A first-order nonlinear hyperbolic system for shallow-water flows

Nonlinear shallow-water (NSW) equations

Let $\Omega \subset \mathbb{R}^2$ be an open domain. We consider the following system:

$$\begin{cases} \partial_t \eta + \nabla_{\mathbf{x}} \cdot \mathbf{q} = 0, \\ \partial_t \mathbf{q} + \nabla_{\mathbf{x}} \cdot \left(\mathbf{u} \otimes \mathbf{q} + \frac{g\eta}{2}(\eta - 2b)\mathbb{I}_2 \right) = -g\eta \nabla_{\mathbf{x}} b. \end{cases}$$

► $b : \Omega \ni \mathbf{x} \mapsto b(\mathbf{x}) \in \mathbb{R}$

Topography

A first-order nonlinear hyperbolic system for shallow-water flows

Nonlinear shallow-water (NSW) equations

Let $\Omega \subset \mathbb{R}^2$ be an open domain. We consider the following system:

$$\begin{cases} \partial_t \eta + \nabla_{\mathbf{x}} \cdot \mathbf{q} = 0, \\ \partial_t \mathbf{q} + \nabla_{\mathbf{x}} \cdot \left(\mathbf{u} \otimes \mathbf{q} + \frac{g\eta}{2}(\eta - 2b)\mathbb{I}_2 \right) = -g\eta \nabla_{\mathbf{x}} b. \end{cases}$$

► $b : \Omega \ni \mathbf{x} \mapsto b(\mathbf{x}) \in \mathbb{R}$

Topography

► $\mathbf{v} : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto (\eta(\mathbf{x}, t), \mathbf{q}(\mathbf{x}, t)) \in \mathcal{H}_b^+$

State vector

A first-order nonlinear hyperbolic system for shallow-water flows

Nonlinear shallow-water (NSW) equations

Let $\Omega \subset \mathbb{R}^2$ be an open domain. We consider the following system:

$$\begin{cases} \partial_t \eta + \nabla_{\mathbf{x}} \cdot \mathbf{q} = 0, \\ \partial_t \mathbf{q} + \nabla_{\mathbf{x}} \cdot \left(\mathbf{u} \otimes \mathbf{q} + \frac{g\eta}{2}(\eta - 2b)\mathbb{I}_2 \right) = -g\eta \nabla_{\mathbf{x}} b. \end{cases}$$

► $b : \Omega \ni \mathbf{x} \mapsto b(\mathbf{x}) \in \mathbb{R}$

Topography

► $\mathbf{v} : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto (\eta(\mathbf{x}, t), \mathbf{q}(\mathbf{x}, t)) \in \mathcal{H}_b^+$

State vector

► $\mathcal{H}_b^+ = \{(\eta, \mathbf{q}) \in \mathbb{R}^3 \mid H := \eta - b > 0\}$

Admissibility set

A first-order nonlinear hyperbolic system for shallow-water flows

Nonlinear shallow-water (NSW) equations

Let $\Omega \subset \mathbb{R}^2$ be an open domain. We consider the following system:

$$\begin{cases} \partial_t \eta + \nabla_{\mathbf{x}} \cdot \mathbf{q} = 0, \\ \partial_t \mathbf{q} + \nabla_{\mathbf{x}} \cdot \left(\mathbf{u} \otimes \mathbf{q} + \frac{g\eta}{2}(\eta - 2b)\mathbb{I}_2 \right) = -g\eta \nabla_{\mathbf{x}} b. \end{cases}$$

► $b : \Omega \ni \mathbf{x} \mapsto b(\mathbf{x}) \in \mathbb{R}$

Topography

► $\mathbf{v} : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto (\eta(\mathbf{x}, t), \mathbf{q}(\mathbf{x}, t)) \in \mathcal{H}_b^+$

State vector

► $\mathcal{H}_b^+ = \{(\eta, \mathbf{q}) \in \mathbb{R}^3 \mid H := \eta - b > 0\}$

Admissibility set

► $\mathbb{F} : \mathcal{H}_b^+ \times \mathbb{R} \ni (\mathbf{v}, b) \mapsto (\mathbf{q}, \mathbf{u} \otimes \mathbf{q} + \frac{g\eta}{2}(\eta - 2b)\mathbb{I}_2)^\top$

Flux tensor

A first-order nonlinear hyperbolic system for shallow-water flows

Nonlinear shallow-water (NSW) equations

Let $\Omega \subset \mathbb{R}^2$ be an open domain. We consider the following system:

$$\begin{cases} \partial_t \eta + \nabla_{\mathbf{x}} \cdot \mathbf{q} = 0, \\ \partial_t \mathbf{q} + \nabla_{\mathbf{x}} \cdot \left(\mathbf{u} \otimes \mathbf{q} + \frac{g\eta}{2}(\eta - 2b)\mathbb{I}_2 \right) = -g\eta \nabla_{\mathbf{x}} b. \end{cases}$$

- ▶ $b : \Omega \ni \mathbf{x} \mapsto b(\mathbf{x}) \in \mathbb{R}$ **Topography**
- ▶ $\mathbf{v} : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto (\eta(\mathbf{x}, t), \mathbf{q}(\mathbf{x}, t)) \in \mathcal{H}_b^+$ **State vector**
- ▶ $\mathcal{H}_b^+ = \{(\eta, \mathbf{q}) \in \mathbb{R}^3 \mid H := \eta - b > 0\}$ **Admissibility set**
- ▶ $\mathbb{F} : \mathcal{H}_b^+ \times \mathbb{R} \ni (\mathbf{v}, b) \mapsto (\mathbf{q}, \mathbf{u} \otimes \mathbf{q} + \frac{g\eta}{2}(\eta - 2b)\mathbb{I}_2)^\top$ **Flux tensor**
- ▶ $\mathbf{B}[b] : \mathcal{H}_b^+ \ni \mathbf{v} \mapsto (0, -g\eta \nabla_{\mathbf{x}} b)^\top$ **Topography source term**

A first-order nonlinear hyperbolic system for shallow-water flows

Nonlinear shallow-water (NSW) equations

Let $\Omega \subset \mathbb{R}^2$ be an open domain. We consider the following system:

$$\begin{cases} \partial_t \eta + \nabla_{\mathbf{x}} \cdot \mathbf{q} = 0, \\ \partial_t \mathbf{q} + \nabla_{\mathbf{x}} \cdot \left(\mathbf{u} \otimes \mathbf{q} + \frac{g\eta}{2}(\eta - 2b)\mathbb{I}_2 \right) = -g\eta \nabla_{\mathbf{x}} b. \end{cases}$$

- ▶ $b : \Omega \ni \mathbf{x} \mapsto b(\mathbf{x}) \in \mathbb{R}$ **Topography**
- ▶ $\mathbf{v} : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto (\eta(\mathbf{x}, t), \mathbf{q}(\mathbf{x}, t)) \in \mathcal{H}_b^+$ **State vector**
- ▶ $\mathcal{H}_b^+ = \{(\eta, \mathbf{q}) \in \mathbb{R}^3 \mid H := \eta - b > 0\}$ **Admissibility set**
- ▶ $\mathbb{F} : \mathcal{H}_b^+ \times \mathbb{R} \ni (\mathbf{v}, b) \mapsto (\mathbf{q}, \mathbf{u} \otimes \mathbf{q} + \frac{g\eta}{2}(\eta - 2b)\mathbb{I}_2)^\top$ **Flux tensor**
- ▶ $B[b] : \mathcal{H}_b^+ \ni \mathbf{v} \mapsto (0, -g\eta \nabla_{\mathbf{x}} b)^\top$ **Topography source term**

Model assumptions

- ▶ It is valid in the **shallow-water regime** → vertical var. are negligible,
- ▶ But it does not account for **dispersive effects** unlike Boussinesq and Serre-Green-Naghdi models.

Table of contents

1. Introduction

2. A new high-order framework for shallow-water asymptotics

Nonlinear shallow-water equations

Classical numerical methods for hyperbolic systems

High-order robust subcell monolithic DG/FV schemes

Arbitrary Lagrangian–Eulerian formulation

3. Waves interacting with floating structures in shallow-water regimes

Physical setting and constraints

Governing equations for water waves

Coupling with a floating object

Numerical scheme

4. Conclusion and perspectives



Slides available online at sachacardonna.github.io

Ideal numerical framework for the NSW system

Two main methods are widely used for hyperbolic systems: **Finite Volume (FV)** and **Discontinuous Galerkin (DG)**. Our method should ideally combine:

Adaptable to source terms

DG: high-order integration ✓

FV: local treatment ✓

Well-balanced

DG: specific treatment 🔧

FV: specific treatment 🔧

High-order accurate

DG: naturally high-order ✓

FV: reconstruction required 🔧

Positivity-preserving

DG: not inherently positive ✗

FV: invariant-domain scheme ✓

Shock-capturing

DG: Gibbs phenomena 🔧

FV: naturally robust ✓

Table of contents

1. Introduction
2. **A new high-order framework for shallow-water asymptotics**
 - Nonlinear shallow-water equations
 - Classical numerical methods for hyperbolic systems
 - High-order robust subcell monolithic DG/FV schemes
 - Arbitrary Lagrangian–Eulerian formulation
3. Waves interacting with floating structures in shallow-water regimes
 - Physical setting and constraints
 - Governing equations for water waves
 - Coupling with a floating object
 - Numerical scheme
4. Conclusion and perspectives



Slides available online at sachacardonna.github.io

Stabilization principle

⚠ High-order schemes need **stabilization** to prevent non-physical oscillations near discontinuities and ensure the robustness of the method.

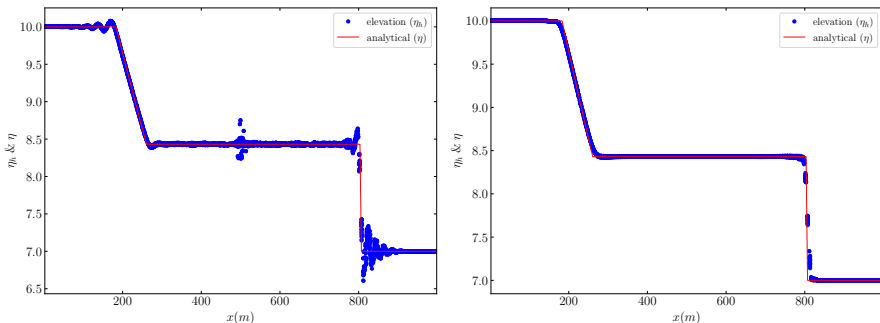
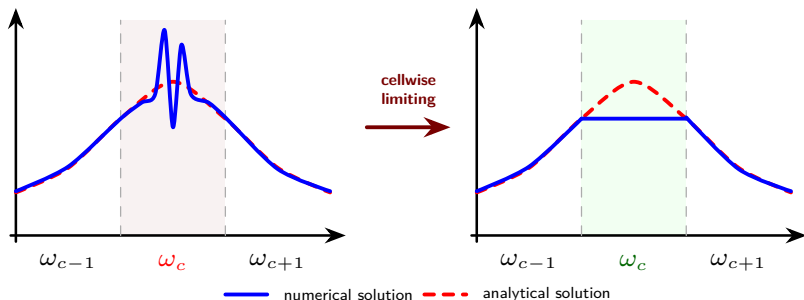


Figure: Dam-break test-case: $\mathbb{P}^5$ pure DG elevation (left) and monolithic DG/FV subcell elevation (right).

Stabilization principle

⚠ High-order schemes need **stabilization** to prevent non-physical oscillations near discontinuities and ensure the robustness of the method.

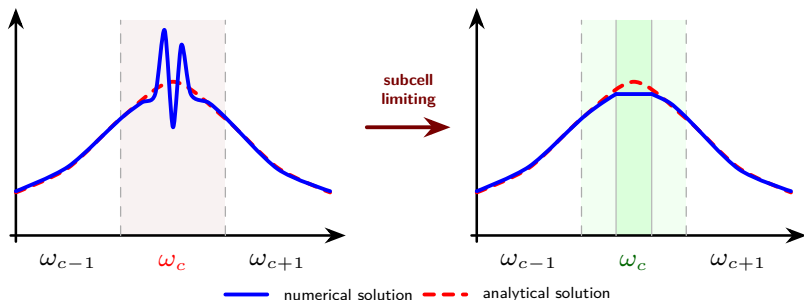
- ▶ **Classical stabilization:** apply limiters/a posteriori correction on the full cell
→ risks **discarding** a mostly accurate solution due to a **local failure**



Stabilization principle

⚠ High-order schemes need **stabilization** to prevent non-physical oscillations near discontinuities and ensure the robustness of the method.

- ▶ **Classical stabilization:** apply limiters/a posteriori correction on the full cell
↪ risks **discarding** a mostly accurate solution due to a **local failure**
- ▶ **Subcell approach:** divide each cell into subcells to reduce the correction scale
→ enabling a **surgical correction** to fix only what's necessary



Stabilization principle

⚠ High-order schemes need **stabilization** to prevent non-physical oscillations near discontinuities and ensure the robustness of the method.

- ▶ **Classical stabilization:** apply limiters/a posteriori correction on the full cell
↪ risks **discarding** a mostly accurate solution due to a **local failure**
- ▶ **Subcell approach:** divide each cell into subcells to reduce the correction scale → enabling a **surgical correction** to fix only what's necessary

Some pioneering references for subcell schemes

📄 **M. Dumbser & R. Loubère**, *A simple robust and accurate a posteriori subcell finite volume limiter for the discontinuous Galerkin method on unstructured meshes*. J. Comp. Phys., 2016.

📄 **F. Vilar**, *A Posteriori Correction of High-Order Discontinuous Galerkin Scheme through Subcell Finite Volume Formulation and Flux Reconstruction*. J. Comp. Phys., 2018.

📄 **A. Rueda-Ramírez, B. Bolm, D. Kuzmin & G. Gassner**, *Monolithic convex limiting for Legendre-Gauss-Lobatto Discontinuous Galerkin Spectral-Element methods*. Commun. Appl. Math., 2024.

A classical mesh ...

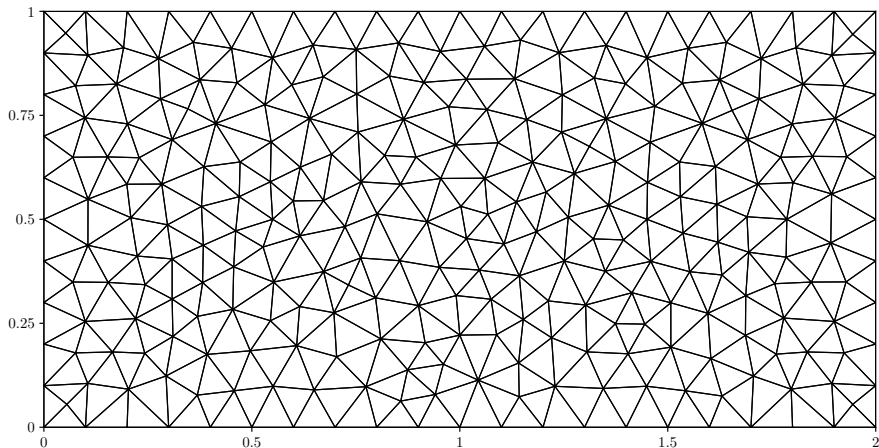


Figure: Unstructured simplicial mesh with $n_{\text{el}} = 350$ cells.

... and its subdivision

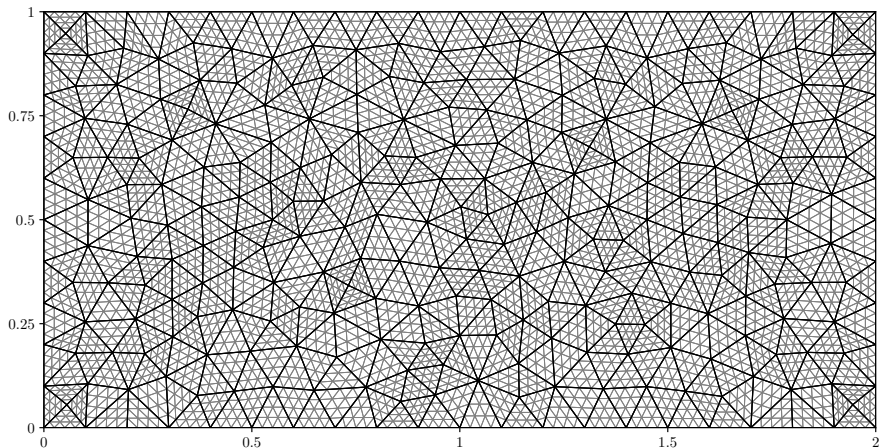


Figure: Unstructured simplicial mesh $\mathbb{P}^3$ subdivision onto triangles with $n_{\text{el}} = 350$ cells.

1st step: reformulation of DG

How to get the best of both worlds between precision and robustness

- ▶ DG is high-order accurate but may oscillate near discontinuities,
 - ▶ FV is robust (in particular positivity-friendly) but low-order,
- ↪ Two complementary methods that can't live on the same framework...

1st step: reformulation of DG

How to get the best of both worlds between precision and robustness

- ▶ DG is high-order accurate but may oscillate near discontinuities,
 - ▶ FV is robust (in particular positivity-friendly) but low-order,
- ↪ Two complementary methods that can't live on the same framework...
- 💡 **Reformulating DG** as a **FV-like subcell scheme**, we will then be able to blend these two approaches in a single framework!

1st step: reformulation of DG

How to get the best of both worlds between precision and robustness

- ▶ DG is high-order accurate but may oscillate near discontinuities,
 - ▶ FV is robust (in particular positivity-friendly) but low-order,
- ↪ Two complementary methods that can't live on the same framework...

💡 **Reformulating DG** as a **FV-like subcell scheme**, we will then be able to blend these two approaches in a single framework!

Key ingredient: subcell viewpoint

Considering v_h^c the DG numerical solution on each cell $\omega_c \subset \Omega$, we build a sub-partition $\{S_m^c\}_{m=1}^{N_s} := \{S_1^c, S_2^c, \dots, S_{N_s}^c\}$ and track subcell averages

$$\bar{v}_m^c(t) = \frac{1}{|S_m^c|} \int_{S_m^c} v_h^c(\mathbf{x}, t) \, d\mathbf{x}$$

📄 **F. Vilar**, *Local subcell monolithic DG/FV convex property preserving scheme on unstructured grids and entropy consideration*. JCP, 2025.

📄 **S.C., F. Marche & F. Vilar**, *A high-order robust subcell monolithic DG/FV formulation for NSW equations on unstructured grids*. JCP (rev.), 2026.

Submean values and polynomial moments

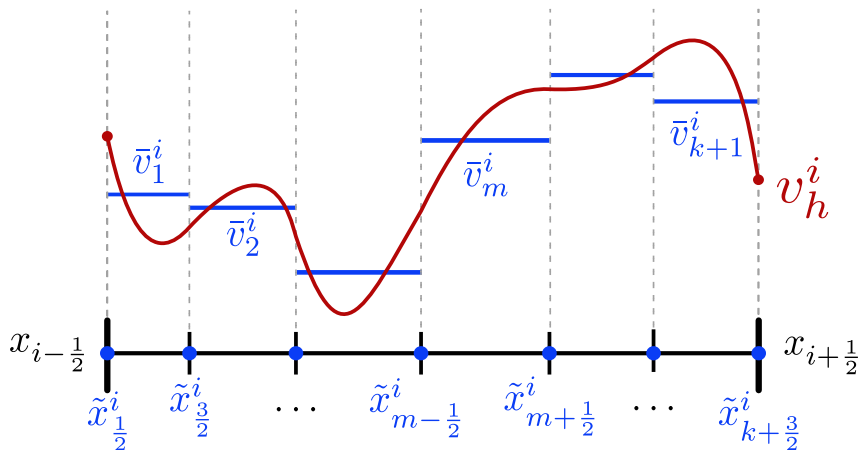
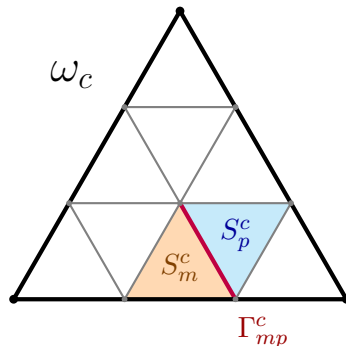


Figure: Piecewise polynomial function v_h^i and associated sub-mean-values (1D case).

2nd step: limiting through flux blending

Blended numerical flux

On each subsurface Γ_{mp}^c of each subcell $S_m^c \in \{S_j^c\}_{j=1}^{N_s}$, we combine two fluxes:



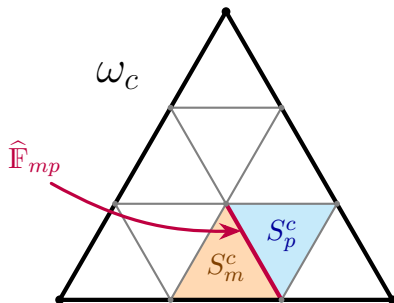
2nd step: limiting through flux blending

Blended numerical flux

On each subsurface Γ_{mp}^c of each subcell $S_m^c \in \{S_j^c\}_{j=1}^{N_s}$, we combine two fluxes:

- A high-order reconstructed **DG flux** $\widehat{\mathbb{F}}_{mp}$, precise but oscillatory,

$$\widetilde{\mathbb{F}}_{mp} = \mathbb{F}_{mp}^{*,FV} + \Theta_{mp} \left(\widehat{\mathbb{F}}_{mp} - \mathbb{F}_{mp}^{*,FV} \right)$$



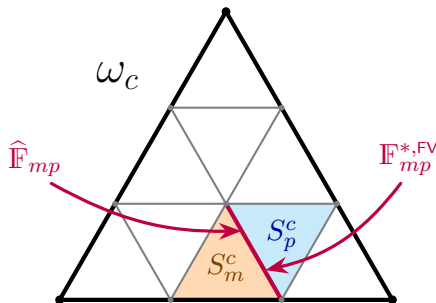
2nd step: limiting through flux blending

Blended numerical flux

On each subsurface Γ_{mp}^c of each subcell $S_m^c \in \{S_j^c\}_{j=1}^{N_s}$, we combine two fluxes:

- ▶ A high-order reconstructed **DG flux** $\widehat{\mathbb{F}}_{mp}$, precise but oscillatory,
- ▶ A first-order domain-invariant **FV flux** $\mathbb{F}_{mp}^{*,FV}$, very robust but less accurate,

$$\widetilde{\mathbb{F}}_{mp} = \mathbb{F}_{mp}^{*,FV} + \Theta_{mp} \left(\widehat{\mathbb{F}}_{mp} - \mathbb{F}_{mp}^{*,FV} \right)$$



2nd step: limiting through flux blending

Blended numerical flux

On each subface Γ_{mp}^c of each subcell $S_m^c \in \{S_j^c\}_{j=1}^{N_s}$, we combine two fluxes:

- ▶ A high-order reconstructed **DG flux** $\hat{\mathbb{F}}_{mp}$, precise but oscillatory,
- ▶ A first-order domain-invariant **FV flux** $\mathbb{F}_{mp}^{*,FV}$, very robust but less accurate,

$$\tilde{\mathbb{F}}_{mp} = \mathbb{F}_{mp}^{*,FV} + \Theta_{mp} \left(\hat{\mathbb{F}}_{mp} - \mathbb{F}_{mp}^{*,FV} \right)$$

⚠ The **blending coefficient** $\Theta_{mp} \in [0, 1]$ is:

- ▶ Computed *a priori* on each Γ_{mp}^c , at each time step (or RK stage),
- ▶ Uniquely defined, i.e. $\Theta_{mp} = \Theta_{pm}$, for all $S_p^v \in \text{Neigh}(S_m^c)$.

↪ Ensures a **discrete maximum principle** and **water height positivity**!

Flux blending and fully-discrete scheme

Blended numerical flux

On each subface Γ_{mp}^c of each subcell $S_m^c \in \{S_j^c\}_{j=1}^{N_s}$, we combine two fluxes:

- ▶ A high-order reconstructed **DG flux** $\hat{\mathbb{F}}_{mp}$, precise but oscillatory,
- ▶ A first-order domain-invariant **FV flux** $\mathbb{F}_{mp}^{*,FV}$, very robust but less accurate,

$$\tilde{\mathbb{F}}_{mp} = \mathbb{F}_{mp}^{*,FV} + \underbrace{\Theta_{mp}}_{=1} \left(\hat{\mathbb{F}}_{mp} - \mathbb{F}_{mp}^{*,FV} \right) = \hat{\mathbb{F}}_{mp}$$

⚠ The **blending coefficient** $\Theta_{mp} \in [0, 1]$ is:

- ▶ Computed *a priori* on each Γ_{mp}^c , at each time step (or RK stage),
- ▶ Uniquely defined, i.e. $\Theta_{mp} = \Theta_{pm}$, for all $S_p^v \in \text{Neigh}(S_m^c)$.

↪ Ensures a **discrete maximum principle** and **water height positivity**!

Flux blending and fully-discrete scheme

Blended numerical flux

On each subface Γ_{mp}^c of each subcell $S_m^c \in \{S_j^c\}_{j=1}^{N_s}$, we combine two fluxes:

- ▶ A high-order reconstructed **DG flux** $\hat{\mathbb{F}}_{mp}$, precise but oscillatory,
- ▶ A first-order domain-invariant **FV flux** $\mathbb{F}_{mp}^{*,FV}$, very robust but less accurate,

$$\tilde{\mathbb{F}}_{mp} = \mathbb{F}_{mp}^{*,FV} + \underbrace{\Theta_{mp}}_{=0} \left(\hat{\mathbb{F}}_{mp} - \mathbb{F}_{mp}^{*,FV} \right) = \mathbb{F}_{mp}^{*,FV}$$

⚠ The **blending coefficient** $\Theta_{mp} \in [0, 1]$ is:

- ▶ Computed *a priori* on each Γ_{mp}^c , at each time step (or RK stage),
- ▶ Uniquely defined, i.e. $\Theta_{mp} = \Theta_{pm}$, for all $S_p^v \in \text{Neigh}(S_m^c)$.

↪ Ensures a **discrete maximum principle** and **water height positivity**!

Flux blending and fully-discrete scheme

Blended numerical flux

On each subface Γ_{mp}^c of each subcell $S_m^c \in \{S_j^c\}_{j=1}^{N_s}$, we combine two fluxes:

- ▶ A high-order reconstructed **DG flux** $\widehat{\mathbb{F}}_{mp}$, precise but oscillatory,
- ▶ A first-order domain-invariant **FV flux** $\mathbb{F}_{mp}^{*,FV}$, very robust but less accurate,

$$\widetilde{\mathbb{F}}_{mp} = \mathbb{F}_{mp}^{*,FV} + \Theta_{mp} \left(\widehat{\mathbb{F}}_{mp} - \mathbb{F}_{mp}^{*,FV} \right)$$

⚠ The **blending coefficient** $\Theta_{mp} \in [0, 1]$ is:

- ▶ Computed *a priori* on each Γ_{mp}^c , at each time step (or RK stage),
- ▶ Uniquely defined, i.e. $\Theta_{mp} = \Theta_{pm}$, for all $S_p^v \in \text{Neigh}(S_m^c)$.

↪ Ensures a **discrete maximum principle** and **water height positivity**!

Flux blending and fully-discrete scheme

Blended numerical flux

On each subface Γ_{mp}^c of each subcell $S_m^c \in \{S_j^c\}_{j=1}^{N_s}$, we combine two fluxes:

- ▶ A high-order reconstructed **DG flux** $\hat{\mathbb{F}}_{mp}$, precise but oscillatory,
- ▶ A first-order domain-invariant **FV flux** $\mathbb{F}_{mp}^{*,FV}$, very robust but less accurate,

$$\tilde{\mathbb{F}}_{mp} = \mathbb{F}_{mp}^{*,FV} + \Theta_{mp} \left(\hat{\mathbb{F}}_{mp} - \mathbb{F}_{mp}^{*,FV} \right)$$

⚠ The **blending coefficient** $\Theta_{mp} \in [0, 1]$ is:

- ▶ Computed *a priori* on each Γ_{mp}^c , at each time step (or RK stage),
 - ▶ Uniquely defined, i.e. $\Theta_{mp} = \Theta_{pm}$, for all $S_p^v \in \text{Neigh}(S_m^c)$.
- ↪ Ensures a **discrete maximum principle** and **water height positivity**!
- ↪ Tuning these numerical fluxes also allows to preserve the **well-balancing**!

Flux blending and fully-discrete scheme

Blended numerical flux

On each subface Γ_{mp}^c of each subcell $S_m^c \in \{S_j^c\}_{j=1}^{N_s}$, we combine two fluxes:

- ▶ A high-order reconstructed **DG flux** $\widehat{\mathbb{F}}_{mp}$, precise but oscillatory,
- ▶ A first-order domain-invariant **FV flux** $\mathbb{F}_{mp}^{*,FV}$, very robust but less accurate,

$$\widetilde{\mathbb{F}}_{mp} = \mathbb{F}_{mp}^{*,FV} + \Theta_{mp} \left(\widehat{\mathbb{F}}_{mp} - \mathbb{F}_{mp}^{*,FV} \right)$$

⚠ The **blending coefficient** $\Theta_{mp} \in [0, 1]$ is:

- ▶ Computed *a priori* on each Γ_{mp}^c , at each time step (or RK stage),
- ▶ Uniquely defined, i.e. $\Theta_{mp} = \Theta_{pm}$, for all $S_p^v \in \text{Neigh}(S_m^c)$.

↪ Ensures a **discrete maximum principle** and **water height positivity**!

↪ Tuning these numerical fluxes also allows to preserve the **well-balancing**!

Monolithic DG/FV subcell scheme with forward Euler time integration

$$\overline{v}_m^{c,n+1} = \overline{v}_m^{c,n} - \frac{\Delta t^n}{|S_m^c|} \sum_{S_p^v \in \text{Neigh}(S_m^c)} |\Gamma_{mp}^c| \widetilde{\mathbb{F}}_{mp} + \frac{\Delta t^n}{|S_m^c|} \int_{S_m^c} B_h^{c,n} \, dx$$

Generalization to algebraic/geometric source terms

Topography and (nonlinear) friction effects

$$\boxed{S[b](\boldsymbol{v}) := \boldsymbol{B}[b](\boldsymbol{v}) + \boldsymbol{R}[b](\boldsymbol{v})}$$

$$\blacktriangleright \boldsymbol{B}[b](\boldsymbol{v}) = (0, -g\eta\nabla_{\boldsymbol{x}}b)^\top \quad \text{Topography source term}$$

$$\blacktriangleright \boldsymbol{R}[b](\boldsymbol{v}) = \begin{cases} (0, -k_f^2 \boldsymbol{q})^\top, & k_f > 0 \\ \left(0, -n_f^2 \frac{\boldsymbol{q} \|\boldsymbol{q}\|}{(\eta - b)^\gamma}\right)^\top, & n_f, \gamma > 0 \end{cases} \quad \begin{array}{l} \text{Linear friction law} \\ \text{Manning friction law} \end{array}$$

❓ Handled the same way as previously → **easily generalizable**

Generalization to algebraic/geometric source terms

Topography and (nonlinear) friction effects

$$S[b](v) := B[b](v) + R[b](v)$$

► $B[b](v) = (0, -g\eta\nabla_x b)^\top$ **Topography source term**

► $R[b](v) = \begin{cases} (0, -k_f^2 \mathbf{q})^\top, & k_f > 0 \\ \left(0, -n_f^2 \frac{\mathbf{q} \|\mathbf{q}\|}{(\eta - b)^\gamma}\right)^\top, & n_f, \gamma > 0 \end{cases}$ **Linear friction law**

Manning friction law

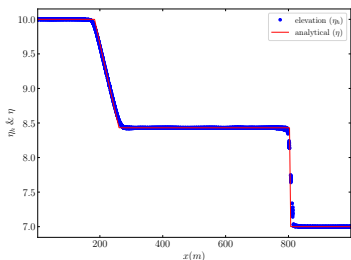
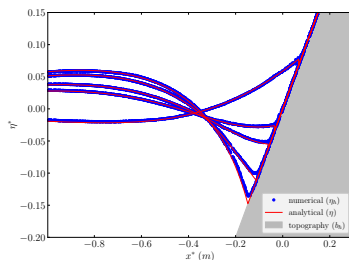
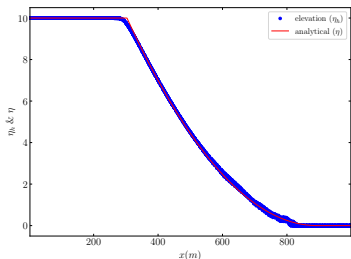
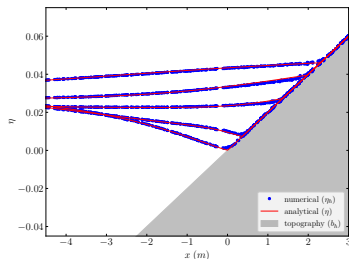
❓ Handled the same way as previously → **easily generalizable**

Applications to Serre–Green–Naghdi (SGN) dispersive equations

Reformulation: Elliptic problem + NSW with dispersive source term

1. Elliptic problem solved *independently*, using a finite element method,
2. Resulting dispersive source term discretized within the NSW framework.

Some numerical results on unstructured meshes with analytical solutions


 $\mathbb{P}^5$ dam-break on a wet bed with $n_{el} = 350$ cells.

 $\mathbb{P}^2$ Carrier-Greenspan periodic sol. with $n_{el} = 1092$ cells.

 $\mathbb{P}^3$ dam-break on a dry bed with $n_{el} = 350$ cells.

 $\mathbb{P}^2$ Carrier-Greenspan transient sol. with $n_{el} = 1092$ cells.

Wave interaction with an island

- **Domain:** $\Omega = [5, 25] \times [0, 30]$ **Degree:** $k = 6$ **Mesh:** $n_{\text{el}} = 584$
- **Goal:** assessing robustness and correct shock-capturing in challenging case

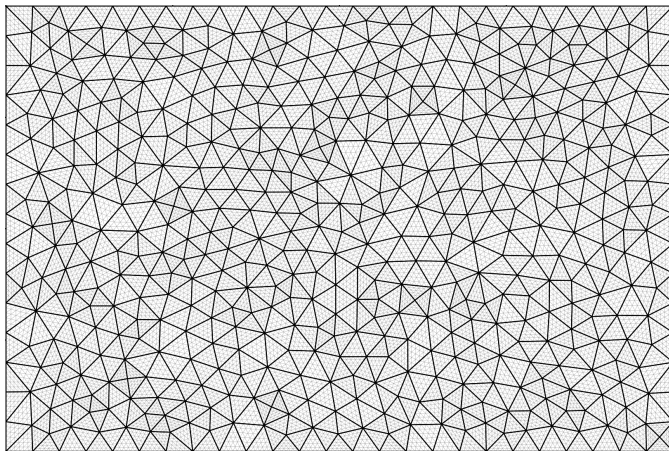


Figure: Unstructured simplicial mesh $\mathbb{P}^6$ subdivision onto triangles with $n_{\text{el}} = 584$ cells → **simulation N°1.**

Table of contents

1. Introduction

2. A new high-order framework for shallow-water asymptotics

Nonlinear shallow-water equations

Classical numerical methods for hyperbolic systems

High-order robust subcell monolithic DG/FV schemes

Arbitrary Lagrangian–Eulerian formulation

3. Waves interacting with floating structures in shallow-water regimes

Physical setting and constraints

Governing equations for water waves

Coupling with a floating object

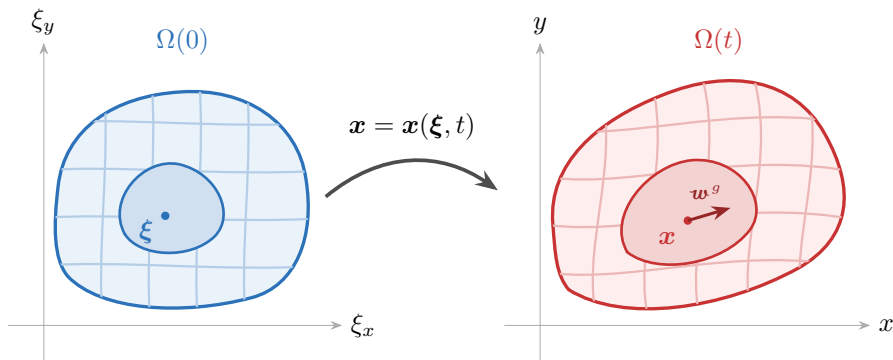
Numerical scheme

4. Conclusion and perspectives



Slides available online at sachacardonna.github.io

What is a direct Arbitrary Lagrangian–Eulerian scheme?



$$\boxed{\Omega(0) \ni \xi \mapsto x := x(\xi, t) \in \Omega(t)}$$

$$\mathbf{w}^g(x, t) = \partial_t \mathbf{x}(\xi, t).$$

↪ Updated at **each RK step** with **no topological change**.

Monolithic DG/FV scheme on moving meshes

Between Eulerian and Lagrangian descriptions

- ▶ **Eulerian:** $\mathbf{w}^g = \mathbf{0}$ → fixed and geometrically robust mesh,
- ▶ **Lagrangian:** $\mathbf{w}^g = \mathbf{u}$ → the mesh follows the fluid but may distort,
- ▶ **ALE:** $\mathbf{w}^g$ is chosen independently → motion where needed.

Monolithic DG/FV scheme on moving meshes

Between Eulerian and Lagrangian descriptions

- ▶ **Eulerian:** $\mathbf{w}^g = \mathbf{0} \rightarrow$ fixed and geometrically robust mesh,
- ▶ **Lagrangian:** $\mathbf{w}^g = \mathbf{u} \rightarrow$ the mesh follows the fluid but may distort,
- ▶ **ALE:** $\mathbf{w}^g$ is chosen independently $\rightarrow$ motion where needed.

The physical flux becomes an ALE flux

$$\mathbb{G}(\mathbf{v}, b, \mathbf{w}^g) = \mathbb{F}(\mathbf{v}, b) - \mathbf{v} \otimes \mathbf{w}^g, \quad \sigma_{\text{ALE}} = |(\mathbf{u} - \mathbf{w}^g) \cdot \mathbf{n}| + \sqrt{gH}.$$

Monolithic DG/FV scheme on moving meshes

Between Eulerian and Lagrangian descriptions

- ▶ **Eulerian:** $\mathbf{w}^g = \mathbf{0} \rightarrow$ fixed and geometrically robust mesh,
- ▶ **Lagrangian:** $\mathbf{w}^g = \mathbf{u} \rightarrow$ the mesh follows the fluid but may distort,
- ▶ **ALE:** $\mathbf{w}^g$ is chosen independently $\rightarrow$ motion where needed.

The physical flux becomes an ALE flux

$$\mathbb{G}(\mathbf{v}, b, \mathbf{w}^g) = \mathbb{F}(\mathbf{v}, b) - \mathbf{v} \otimes \mathbf{w}^g, \quad \sigma_{\text{ALE}} = |(\mathbf{u} - \mathbf{w}^g) \cdot \mathbf{n}| + \sqrt{gH}.$$

The same local subcell blending mechanism

We still combine high-order **DG-ALE flux** with a first-order **FV-ALE flux**
 $\hookrightarrow$ Everything (almost) works the same way as in the Eulerian case!

Monolithic DG/FV scheme on moving meshes

Between Eulerian and Lagrangian descriptions

- ▶ **Eulerian:** $\mathbf{w}^g = \mathbf{0} \rightarrow$ fixed and geometrically robust mesh,
- ▶ **Lagrangian:** $\mathbf{w}^g = \mathbf{u} \rightarrow$ the mesh follows the fluid but may distort,
- ▶ **ALE:** $\mathbf{w}^g$ is chosen independently $\rightarrow$ motion where needed.

The physical flux becomes an ALE flux

$$\mathbb{G}(\mathbf{v}, b, \mathbf{w}^g) = \mathbb{F}(\mathbf{v}, b) - \mathbf{v} \otimes \mathbf{w}^g, \quad \sigma_{\text{ALE}} = |(\mathbf{u} - \mathbf{w}^g) \cdot \mathbf{n}| + \sqrt{gH}.$$

The same local subcell blending mechanism

We still combine high-order **DG-ALE flux** with a first-order **FV-ALE flux**
 $\hookrightarrow$ Everything (almost) works the same way as in the Eulerian case!

Monolithic DG/FV-ALE subcell scheme with forward Euler time integration

$$|S_m^{c,n+1}| \bar{\mathbf{v}}_m^{c,n+1} = |S_m^{c,n}| \bar{\mathbf{v}}_m^{c,n} - \Delta t^n \sum_{S_p^{v,n} \in \text{Neigh}(S_m^c)} |\Gamma_{mp}^{c,n}| \tilde{\mathbb{G}}_{mp} + \Delta t^n \bar{\mathbf{B}}_m^{c,n}$$

Extending a boundary motion to the whole mesh

An elliptic problem for the mesh motion

If one wants to move the mesh according to a prescribed motion of the boundary $\partial\Omega_{\text{mov}}(t)$, one can solve an **elliptic problem** to extend this motion smoothly to the whole domain $\Omega(t)$ to preserve mesh quality:

$$\begin{cases} -\nabla_{\mathbf{x}} \cdot (2\mu(\mathbf{x})\varepsilon(\mathbf{w}^g) + \lambda(\mathbf{x})(\nabla_{\mathbf{x}} \cdot \mathbf{w}^g)\mathbb{I}_2) = \mathbf{0} & \text{in } \Omega(t), \\ \varepsilon(\mathbf{w}^g) := \frac{1}{2} (\nabla_{\mathbf{x}} \mathbf{w}^g + (\nabla_{\mathbf{x}} \mathbf{w}^g)^\top) & \text{in } \Omega(t), \\ \mathbf{w}^g = \mathbf{0} \quad \text{on } \partial\Omega_{\text{ext}} \quad \text{and} \quad \mathbf{w}^g = \partial_t \mathbf{x}_{\text{mov}}(t) & \text{on } \partial\Omega_{\text{mov}}(t). \end{cases}$$

↪ **Elasticity-like deformation** is better suited for large deformations than a simple Laplace problem.

Extending a boundary motion to the whole mesh

An elliptic problem for the mesh motion

If one wants to move the mesh according to a prescribed motion of the boundary $\partial\Omega_{\text{mov}}(t)$, one can solve an **elliptic problem** to extend this motion smoothly to the whole domain $\Omega(t)$ to preserve mesh quality:

$$\begin{cases} -\nabla_{\mathbf{x}} \cdot (2\mu(\mathbf{x})\varepsilon(\mathbf{w}^g) + \lambda(\mathbf{x})(\nabla_{\mathbf{x}} \cdot \mathbf{w}^g)\mathbb{I}_2) = \mathbf{0} & \text{in } \Omega(t), \\ \varepsilon(\mathbf{w}^g) := \frac{1}{2} (\nabla_{\mathbf{x}} \mathbf{w}^g + (\nabla_{\mathbf{x}} \mathbf{w}^g)^\top) & \text{in } \Omega(t), \\ \mathbf{w}^g = \mathbf{0} \quad \text{on } \partial\Omega_{\text{ext}} \quad \text{and} \quad \mathbf{w}^g = \partial_t \mathbf{x}_{\text{mov}}(t) & \text{on } \partial\Omega_{\text{mov}}(t). \end{cases}$$

↪ **Elasticity-like deformation** is better suited for large deformations than a simple Laplace problem.

Finite element discretization

The mesh velocity $\mathbf{w}^g$ is computed at each vertex of the mesh, so we use a **vector-valued continuous $\mathbb{P}^1$ finite element method**. We seek $\mathbf{w}_h^g \in \mathbb{P}^1(\Omega(t))$ such that for all $\mathbf{z}_h \in \mathbb{P}^1(\Omega(t))$:

$$\int_{\Omega(t)} 2\mu(\mathbf{x})\varepsilon(\mathbf{w}_h^g) : \varepsilon(\mathbf{z}_h) \, d\mathbf{x} + \int_{\Omega(t)} \lambda(\mathbf{x})(\nabla_{\mathbf{x}} \cdot \mathbf{w}_h^g)(\nabla_{\mathbf{x}} \cdot \mathbf{z}_h) \, d\mathbf{x} = 0.$$

Some simulations with moving boundaries

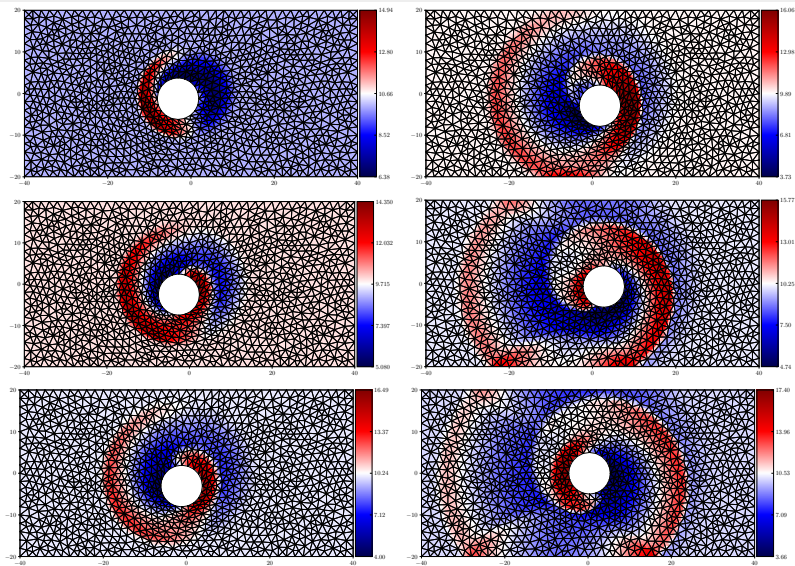


Figure: Snapshots of an oscillating cylindrical wavemaker → **simulation N°2–4.**

Table of contents

1. Introduction

2. A new high-order framework for shallow-water asymptotics

Nonlinear shallow-water equations

Classical numerical methods for hyperbolic systems

High-order robust subcell monolithic DG/FV schemes

Arbitrary Lagrangian–Eulerian formulation

3. Waves interacting with floating structures in shallow-water regimes

Physical setting and constraints

Governing equations for water waves

Coupling with a floating object

Numerical scheme

4. Conclusion and perspectives



Slides available online at sachacardonna.github.io

Table of contents

1. Introduction

2. A new high-order framework for shallow-water asymptotics

Nonlinear shallow-water equations

Classical numerical methods for hyperbolic systems

High-order robust subcell monolithic DG/FV schemes

Arbitrary Lagrangian–Eulerian formulation

3. Waves interacting with floating structures in shallow-water regimes

Physical setting and constraints

Governing equations for water waves

Coupling with a floating object

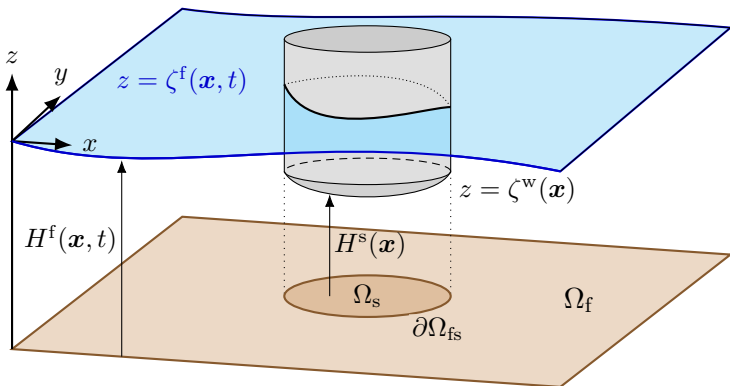
Numerical scheme

4. Conclusion and perspectives



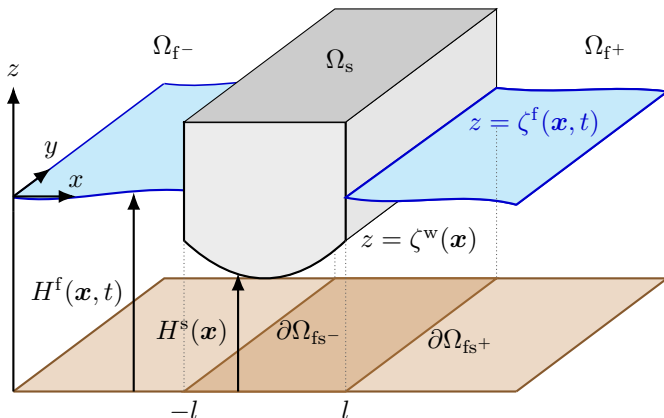
Slides available online at sachacardonna.github.io

3D setting: cylinder structure



- ▶ Fluid domain bounded below by a flat bottom and above by a free surface $z = \zeta(\mathbf{x}, t)$,
- ▶ A rigid, **stationnary** (for the sake of simplicity), partially immersed floating structure interacts with the surrounding flow,
- ▶ Vertical variations are averaged, leading to shallow-water type models.

3D setting: infinite pontoon structure



- ▶ The interior region is an infinite strip $\Omega_s = (-l, l) \times \mathbb{R}$, while the exterior domain consists of two half-planes $\Omega_{f\pm} = \{(x, y) \mid \pm x > l\}$.
- ▶ This configuration provides an **excellent benchmark**, since several quantities can be derived analytically, making it valuable for validation.

2D horizontal setting

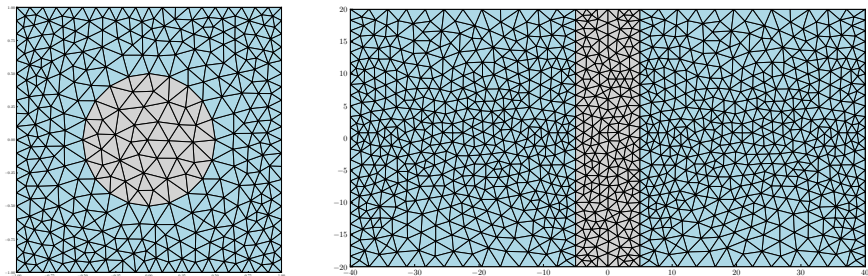


Figure: Computational meshes for a cylinder object (left) and a pontoon-like object (right).

For the latter numerical resolution, we work in a 2D horizontal setting, where the domain is decomposed into three parts:

- ▶ Ω_s (“solid region”): hor. projection of the wet part under the object,
- ▶ Ω_f (“fluid region”): hor. projection of the free surface in contact with air,
- ▶ $\partial\Omega_{fs}$: interface separating them, with unit normal $\mathbf{n}$ pointing toward Ω_f .

Notation: we will denote with a superscript $\bullet^s$ (resp. $\bullet^f$) the restriction of a quantity to Ω_s (resp. Ω_f).

Table of contents

1. Introduction
2. A new high-order framework for shallow-water asymptotics
 - Nonlinear shallow-water equations
 - Classical numerical methods for hyperbolic systems
 - High-order robust subcell monolithic DG/FV schemes
 - Arbitrary Lagrangian–Eulerian formulation
3. **Waves interacting with floating structures in shallow-water regimes**
 - Physical setting and constraints
 - Governing equations for water waves
 - Coupling with a floating object
 - Numerical scheme
4. Conclusion and perspectives



Slides available online at sachacardonna.github.io

Baseline model for water waves

An alternative form of NSW equations

$$\begin{cases} \partial_t \zeta^f + \nabla_{\mathbf{x}} \cdot (H^f \mathbf{u}^f) = 0 \\ \partial_t \mathbf{u}^f + \nabla_{\mathbf{x}} (g\zeta^f + \frac{1}{2}|\mathbf{u}^f|^2) = -\frac{1}{\rho} \nabla_{\mathbf{x}} \underline{p}^f \end{cases}$$

- ▶ $\zeta^f : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto \zeta^f(\mathbf{x}, t)$ **elevation w.r.t. rest state**
- ▶ $\mathbf{u}^f : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto \mathbf{u}^f(\mathbf{x}, t)$ **horizontal velocity**
- ▶ $H^f : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto H^f(\mathbf{x}, t) := H_0 + \zeta^f(\mathbf{x}, t)$ **water height**
- ▶ $\underline{p}^f : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto \underline{p}^f(\mathbf{x}, t)$ **pressure**

Baseline model for water waves

An alternative form of NSW equations

$$\begin{cases} \partial_t \zeta^f + \nabla_{\mathbf{x}} \cdot (H^f \mathbf{u}^f) = 0 \\ \partial_t \mathbf{u}^f + \nabla_{\mathbf{x}} (g\zeta^f + \frac{1}{2}|\mathbf{u}^f|^2) = -\frac{1}{\rho} \nabla_{\mathbf{x}} \underline{p}^f \end{cases}$$

- ▶ $\zeta^f : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto \zeta^f(\mathbf{x}, t)$ **elevation w.r.t. rest state**
- ▶ $\mathbf{u}^f : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto \mathbf{u}^f(\mathbf{x}, t)$ **horizontal velocity**
- ▶ $H^f : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto H^f(\mathbf{x}, t) := H_0 + \zeta^f(\mathbf{x}, t)$ **water height**
- ▶ $\underline{p}^f : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto \underline{p}^f(\mathbf{x}, t)$ **pressure**

⚠ This **alternative form** ($\zeta^f, \mathbf{u}^f$) is used to derive the model used later
 ⇨ Numerically we use the previous system with variables $(\eta^f, \mathbf{q}^f)$ instead.

Baseline model for water waves

An alternative form of NSW equations

$$\begin{cases} \partial_t \zeta^f + \nabla_{\mathbf{x}} \cdot (H^f \mathbf{u}^f) = 0 \\ \partial_t \mathbf{u}^f + \nabla_{\mathbf{x}} (g\zeta^f + \frac{1}{2}|\mathbf{u}^f|^2) = -\frac{1}{\rho} \nabla_{\mathbf{x}} \underline{p}^f \end{cases}$$

- ▶ $\zeta^f : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto \zeta^f(\mathbf{x}, t)$ **elevation w.r.t. rest state**
- ▶ $\mathbf{u}^f : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto \mathbf{u}^f(\mathbf{x}, t)$ **horizontal velocity**
- ▶ $H^f : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto H^f(\mathbf{x}, t) := H_0 + \zeta^f(\mathbf{x}, t)$ **water height**
- ▶ $\underline{p}^f : \Omega \times (0, T) \ni (\mathbf{x}, t) \mapsto \underline{p}^f(\mathbf{x}, t)$ **pressure**

⚠ This **alternative form** ($\zeta^f, \mathbf{u}^f$) is used to derive the model used later
 ⇨ Numerically we use the previous system with variables $(\eta^f, \mathbf{q}^f)$ instead.

What about the topography source term?

💡 For simplicity, we **neglect topography** as it does not modify the coupling structure nor the transmission conditions, since this source term is purely **geometric** → can be reintroduced later in the numerical experiments!

Table of contents

1. Introduction
2. A new high-order framework for shallow-water asymptotics
 - Nonlinear shallow-water equations
 - Classical numerical methods for hyperbolic systems
 - High-order robust subcell monolithic DG/FV schemes
 - Arbitrary Lagrangian–Eulerian formulation
3. **Waves interacting with floating structures in shallow-water regimes**
 - Physical setting and constraints
 - Governing equations for water waves
 - Coupling with a floating object
 - Numerical scheme
4. Conclusion and perspectives



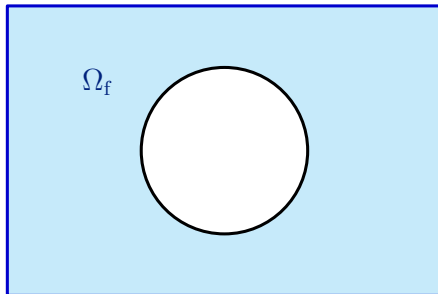
Slides available online at sachacardonna.github.io

Constraints induced by the floating object

Exterior region Ω_f (free surface)

- ▶ Surface is in contact with air,
- ▶ Pressure prescr.: $\underline{p} = p_{\text{atm}}$,
- ▶ The NSW momentum equation has no source term:

$$\partial_t \mathbf{u}^f + \nabla_x \left(g\zeta^f + \frac{1}{2} |\mathbf{u}^f|^2 \right) = \mathbf{0}.$$



Constraints induced by the floating object

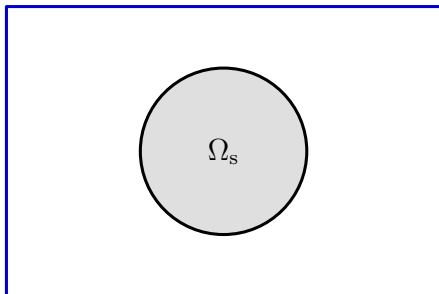
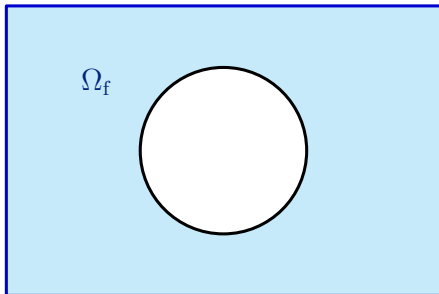
Exterior region Ω_f (free surface)

- ▶ Surface is in contact with air,
- ▶ Pressure prescr.: $\underline{p} = p_{\text{atm}}$,
- ▶ The NSW momentum equation has no source term:

$$\partial_t \mathbf{u}^f + \nabla_x (g\zeta^f + \frac{1}{2}|\mathbf{u}^f|^2) = \mathbf{0}.$$

Interior region Ω_s (under the object)

- ▶ Free surface is blocked by the object (underwater part of the structure),
- ▶ Elevation is prescribed: $\zeta^s = \zeta^w$,
- ▶ Leads to an “incompressible” constraint: $\nabla_x \cdot (H^s \mathbf{u}^s) = 0$.



Constraints induced by the floating object

Exterior region Ω_f (free surface)

- ▶ Surface is in contact with air,
- ▶ Pressure prescr.: $\underline{p} = p_{\text{atm}}$,
- ▶ The NSW momentum equation has no source term:

$$\partial_t \mathbf{u}^f + \nabla_x (g\zeta^f + \frac{1}{2}|\mathbf{u}^f|^2) = \mathbf{0}.$$

Interior region Ω_s (under the object)

- ▶ Free surface is blocked by the object (underwater part of the structure),
- ▶ Elevation is prescribed: $\zeta^s = \zeta^w$,
- ▶ Leads to an “incompressible” constraint: $\nabla_x \cdot (H^s \mathbf{u}^s) = 0$.

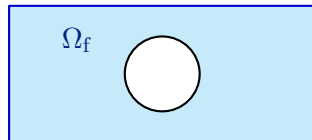
Coupling at the interface $\partial\Omega_{fs}$

- ▶ Mass conservation (normal flux continuity): $H^f \mathbf{u}^f \cdot \mathbf{n} = H^s \mathbf{u}^s \cdot \mathbf{n}$,
- ▶ Bernouilli pressure transmission (energy consistency): $\Pi^f = \Pi^s$,
where $\Pi^f = \rho g \zeta + \frac{1}{2} \rho |\mathbf{u}|^2$ and $\Pi^s = \underline{p}^s - p_{\text{atm}} + \rho g \zeta + \frac{1}{2} \rho |\mathbf{u}|^2$,
- ▶ Velocity compatibility (no artificial vortex): $\mathbf{u}^f \cdot \mathbf{n}^\perp = \mathbf{u}^s \cdot \mathbf{n}^\perp$.

Summary: a coupled “partly constrained” SW system

► **Exterior region Ω_f** (free surface):

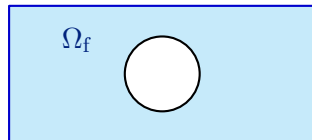
$$\begin{aligned}\partial_t \zeta^f + \nabla_{\mathbf{x}} \cdot (H^f \mathbf{u}^f) &= 0 \\ \partial_t \mathbf{u}^f + \nabla_{\mathbf{x}} \left(g \zeta^f + \frac{1}{2} |\mathbf{u}^f|^2 \right) &= \mathbf{0}\end{aligned}$$



Summary: a coupled “partly constrained” SW system

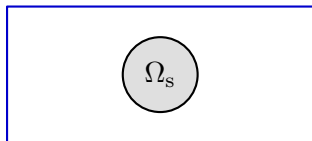
► Exterior region Ω_f (free surface):

$$\begin{aligned}\partial_t \zeta^f + \nabla_{\mathbf{x}} \cdot (H^f \mathbf{u}^f) &= 0 \\ \partial_t \mathbf{u}^f + \nabla_{\mathbf{x}} \left(g \zeta^f + \frac{1}{2} |\mathbf{u}^f|^2 \right) &= \mathbf{0}\end{aligned}$$



► Interior region Ω_s (under the object):

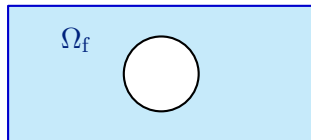
$$\begin{aligned}\nabla_{\mathbf{x}} \cdot (H^s \mathbf{u}^s) &= 0 \\ \partial_t \mathbf{u}^s + \mathbf{u}^s \cdot \nabla_{\mathbf{x}} \mathbf{u}^s + g \nabla_{\mathbf{x}} \zeta^w &= \rho^{-1} \nabla_{\mathbf{x}} \underline{\mathbf{p}}^s\end{aligned}$$



Summary: a coupled “partly constrained” SW system

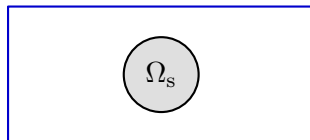
► Exterior region Ω_f (free surface):

$$\begin{aligned}\partial_t \zeta^f + \nabla_{\mathbf{x}} \cdot (H^f \mathbf{u}^f) &= 0 \\ \partial_t \mathbf{u}^f + \nabla_{\mathbf{x}} \left(g \zeta^f + \frac{1}{2} |\mathbf{u}^f|^2 \right) &= \mathbf{0}\end{aligned}$$



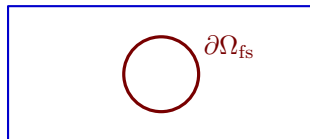
► Interior region Ω_s (under the object):

$$\begin{aligned}\nabla_{\mathbf{x}} \cdot (H^s \mathbf{u}^s) &= 0 \\ \partial_t \mathbf{u}^s + \mathbf{u}^s \cdot \nabla_{\mathbf{x}} \mathbf{u}^s + g \nabla_{\mathbf{x}} \zeta^w &= \rho^{-1} \nabla_{\mathbf{x}} \underline{p}^s\end{aligned}$$



► Interface $\partial\Omega_{fs}$ (coupling conditions):

$$\begin{aligned}H^f \mathbf{u}^f \cdot \mathbf{n} &= H^s \mathbf{u}^s \cdot \mathbf{n} \\ \Pi^f &= \Pi^s \\ \mathbf{u}^f \cdot \mathbf{n}^\perp &= \mathbf{u}^s \cdot \mathbf{n}^\perp\end{aligned}$$



Irrotational initial data: a key simplification

Propagation of irrotationality

If $\nabla_{\mathbf{x}}^{\perp} \cdot \mathbf{u}^f(\cdot, 0) = 0$ in Ω_f and $\nabla_{\mathbf{x}}^{\perp} \cdot \mathbf{u}^s(\cdot, 0) = 0$ in Ω_s , it stays true for all $t \geq 0$
↪ Physically the flow remains smooth, without rotation or vortex generation.

Irrotational initial data: a key simplification

Propagation of irrotationality

If $\nabla_{\mathbf{x}}^{\perp} \cdot \mathbf{u}^f(\cdot, 0) = 0$ in Ω_f and $\nabla_{\mathbf{x}}^{\perp} \cdot \mathbf{u}^s(\cdot, 0) = 0$ in Ω_s , it stays true for all $t \geq 0$
↪ Physically the flow remains smooth, without rotation or vortex generation.

Consequence: the interior becomes elliptic

- ▶ Because the flow remains **irrotational**, the interior velocity can be written as a **gradient field** i.e. there exists a potential ϕ^s such that

$$\mathbf{u}^s = \nabla_{\mathbf{x}} \phi^s \quad \text{in } \Omega_s.$$

Irrotational initial data: a key simplification

Propagation of irrotationality

If $\nabla_{\mathbf{x}}^{\perp} \cdot \mathbf{u}^f(\cdot, 0) = 0$ in Ω_f and $\nabla_{\mathbf{x}}^{\perp} \cdot \mathbf{u}^s(\cdot, 0) = 0$ in Ω_s , it stays true for all $t \geq 0$
↪ Physically the flow remains smooth, without rotation or vortex generation.

Consequence: the interior becomes elliptic

- ▶ Because the flow remains **irrotational**, the interior velocity can be written as a **gradient field** i.e. there exists a potential ϕ^s such that

$$\mathbf{u}^s = \nabla_{\mathbf{x}} \phi^s \quad \text{in } \Omega_s.$$

- ▶ Under the object, the free surface is fixed, hence the water depth is **constant in time**: $\partial_t H^s = \partial_t (H_0 + \zeta^w) = 0$.

Irrotational initial data: a key simplification

Propagation of irrotationality

If $\nabla_{\mathbf{x}}^{\perp} \cdot \mathbf{u}^f(\cdot, 0) = 0$ in Ω_f and $\nabla_{\mathbf{x}}^{\perp} \cdot \mathbf{u}^s(\cdot, 0) = 0$ in Ω_s , it stays true for all $t \geq 0$
 $\hookrightarrow$ Physically the flow remains smooth, without rotation or vortex generation.

Consequence: the interior becomes elliptic

- ▶ Because the flow remains **irrotational**, the interior velocity can be written as a **gradient field** i.e. there exists a potential ϕ^s such that

$$\mathbf{u}^s = \nabla_{\mathbf{x}} \phi^s \quad \text{in } \Omega_s.$$

- ▶ Under the object, the free surface is fixed, hence the water depth is **constant in time**: $\partial_t H^s = \partial_t (H_0 + \zeta^w) = 0$.
- ▶ Combining irrotationality with the incompressibility constraint $\nabla_{\mathbf{x}} \cdot (H^s \mathbf{u}^s) = 0$ leads to an **elliptic problem** for the potential:

$$\nabla_{\mathbf{x}} \cdot (H^s \nabla_{\mathbf{x}} \phi^s) = 0 \quad \text{in } \Omega_s$$

From the interior constraint to a mixed formulation

What happens under the object

- To solve $\nabla_{\mathbf{x}} \cdot (H^s \nabla_{\mathbf{x}} \phi^s) = 0$ in Ω_s , we prescribe the trace of the potential on the interface,

$$\phi^s = \psi^s \quad \text{on } \partial\Omega_{fs}.$$

From the interior constraint to a mixed formulation

What happens under the object

- ▶ To solve $\nabla_{\mathbf{x}} \cdot (H^s \nabla_{\mathbf{x}} \phi^s) = 0$ in Ω_s , we prescribe the trace of the potential on the interface,

$$\phi^s = \psi^s \quad \text{on } \partial\Omega_{fs}.$$

- ▶ The trace ψ^s is **not arbitrary**: it evolves in time according to an ODE obtained from the Bernoulli relation on the interface i.e.

$$\partial_t \psi^s = -g\zeta - \frac{1}{2}|\mathbf{u}|^2 \quad \text{on } \partial\Omega_{fs}.$$

From the interior constraint to a mixed formulation

What happens under the object

- ▶ To solve $\nabla_{\mathbf{x}} \cdot (H^s \nabla_{\mathbf{x}} \phi^s) = 0$ in Ω_s , we prescribe the trace of the potential on the interface,

$$\phi^s = \psi^s \quad \text{on } \partial\Omega_{fs}.$$

- ▶ The trace ψ^s is **not arbitrary**: it evolves in time according to an ODE obtained from the Bernoulli relation on the interface i.e.

$$\partial_t \psi^s = -g\zeta - \frac{1}{2}|\mathbf{u}|^2 \quad \text{on } \partial\Omega_{fs}.$$

How the interior talks to the exterior

We define the following **Dirichlet-to-Neumann (DtN) operator**:

$$\Lambda \psi^s := H^s \nabla_{\mathbf{x}} \phi^s \cdot \mathbf{n} \quad \text{on } \partial\Omega_{fs}$$

maps the potential ψ^s (Dirichlet data) to the normal flux (Neumann data)

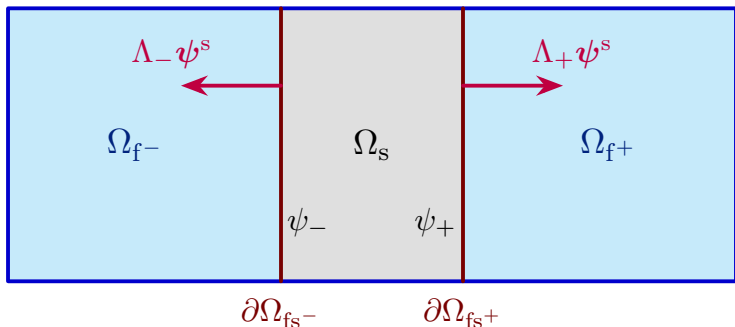
↪ Quantifies how the interior motion exchanges water with the exterior flow

Digression: the infinite pontoon case

Infinite pontoon and generalized DtN operator

We prescribe the traces of the interior potential, i.e. $\phi^s = \psi_{\pm}$ on $\partial\Omega_{fs\pm} := \{(\pm l, y) \mid y \in \mathbb{R}\}$ and for $\psi^s = (\psi_-, \psi_+)$ define

$$\Lambda\psi^s = (\Lambda_-\psi^s, \Lambda_+\psi^s), \quad \Lambda_{\pm}\psi^s = \pm H^s(\partial_x\phi^s)|_{x=\pm l}.$$



Digression: the infinite pontoon case

Infinite pontoon and generalized DtN operator

We prescribe the traces of the interior potential, i.e. $\phi^s = \psi_{\pm}$ on $\partial\Omega_{fs^{\pm}} := \{(\pm l, y) \mid y \in \mathbb{R}\}$ and for $\psi^s = (\psi_-, \psi_+)$ define

$$\Lambda \psi^s = (\Lambda_- \psi^s, \Lambda_+ \psi^s), \quad \Lambda_{\pm} \psi^s = \pm H^s(\partial_x \phi^s)|_{x=\pm l}.$$

Why this configuration matters?

Because the **DtN operator** can then be written **explicitly** as:

$$\begin{aligned} \Lambda_- \psi^s(y) &= H^s \mathfrak{F}^{-1} \left(\xi \coth(2l\xi) \widehat{\psi}_-(\xi) \right) (y) - H^s \mathfrak{F}^{-1} \left(\frac{\xi}{\sinh(2l\xi)} \widehat{\psi}_+(\xi) \right) (y), \\ \Lambda_+ \psi^s(y) &= -H^s \mathfrak{F}^{-1} \left(\frac{\xi}{\sinh(2l\xi)} \widehat{\psi}_-(\xi) \right) (y) + H^s \mathfrak{F}^{-1} \left(\xi \coth(2l\xi) \widehat{\psi}_+(\xi) \right) (y). \end{aligned}$$

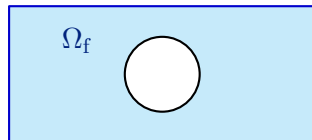
where $\widehat{\psi}_{\pm} = \mathfrak{F}(\psi_{\pm})$ denotes the Fourier transform with respect to the transverse variable y .

↪ No **elliptic solver** is needed inside the structure, which makes an ideal benchmark to validate the coupling strategy **independently** of the interior solver!

Equivalent mixed formulation (irrotational case)

► **Exterior region Ω_f** (free surface):

$$\begin{aligned}\partial_t \zeta^f + \nabla_{\mathbf{x}} \cdot (H^f \mathbf{u}^f) &= 0, \\ \partial_t \mathbf{u}^f + \nabla_{\mathbf{x}} \left(g \zeta^f + \frac{1}{2} |\mathbf{u}^f|^2 \right) &= \mathbf{0}\end{aligned}$$



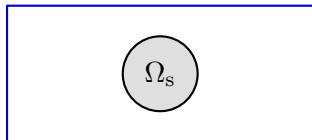
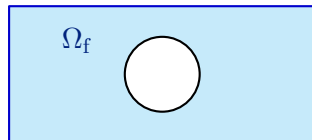
Equivalent mixed formulation (irrotational case)

► **Exterior region Ω_f** (free surface):

$$\begin{aligned}\partial_t \zeta^f + \nabla_{\mathbf{x}} \cdot (H^f \mathbf{u}^f) &= 0, \\ \partial_t \mathbf{u}^f + \nabla_{\mathbf{x}} \left(g \zeta^f + \frac{1}{2} |\mathbf{u}^f|^2 \right) &= \mathbf{0}\end{aligned}$$

► **Interior region Ω_s** (under the object):

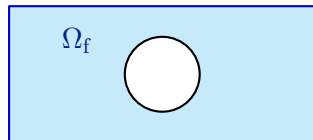
$$\begin{aligned}\nabla_{\mathbf{x}} \cdot (H^s \nabla_{\mathbf{x}} \phi^s) &= 0 \quad \text{in } \Omega_s \\ \phi^s &= \psi^s \quad \text{on } \partial\Omega_{fs}\end{aligned}$$



Equivalent mixed formulation (irrotational case)

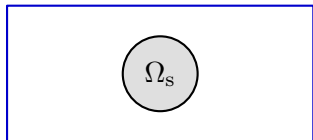
► Exterior region Ω_f (free surface):

$$\begin{aligned}\partial_t \zeta^f + \nabla_{\mathbf{x}} \cdot (H^f \mathbf{u}^f) &= 0, \\ \partial_t \mathbf{u}^f + \nabla_{\mathbf{x}} \left(g \zeta^f + \frac{1}{2} |\mathbf{u}^f|^2 \right) &= \mathbf{0}\end{aligned}$$



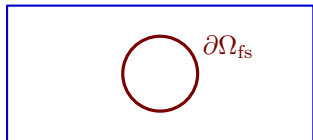
► Interior region Ω_s (under the object):

$$\begin{aligned}\nabla_{\mathbf{x}} \cdot (H^s \nabla_{\mathbf{x}} \phi^s) &= 0 \quad \text{in } \Omega_s \\ \phi^s &= \psi^s \quad \text{on } \partial\Omega_{fs}\end{aligned}$$



► Interface $\partial\Omega_{fs}$ (boundary coupling):

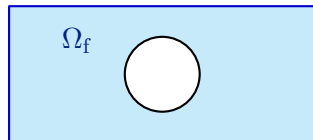
$$\begin{aligned}\mathbf{n} \cdot (H \mathbf{u}) &= \Lambda \psi^s \\ \partial_t \psi^s &= -g \zeta - \frac{1}{2} |\mathbf{u}|^2\end{aligned}$$



Equivalent mixed formulation (irrotational case)

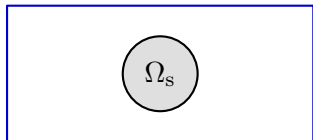
► Exterior region Ω_f (free surface):

$$\begin{aligned}\partial_t \zeta^f + \nabla_{\mathbf{x}} \cdot (H^f \mathbf{u}^f) &= 0, \\ \partial_t \mathbf{u}^f + \nabla_{\mathbf{x}} \left(g \zeta^f + \frac{1}{2} |\mathbf{u}^f|^2 \right) &= \mathbf{0}\end{aligned}$$



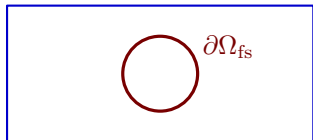
► Interior region Ω_s (under the object):

$$\begin{aligned}\nabla_{\mathbf{x}} \cdot (H^s \nabla_{\mathbf{x}} \phi^s) &= 0 \quad \text{in } \Omega_s \\ \phi^s &= \psi^s \quad \text{on } \partial\Omega_{fs}\end{aligned}$$



► Interface $\partial\Omega_{fs}$ (boundary coupling):

$$\begin{aligned}\mathbf{n} \cdot (H \mathbf{u}) &= \Lambda \psi^s \\ \partial_t \psi^s &= -g \zeta - \frac{1}{2} |\mathbf{u}|^2\end{aligned}$$



► Initial conditions: $(\zeta^f, \mathbf{u}^f)|_{t=0} = (\zeta^{\text{in}}, \mathbf{u}^{\text{in}})$ in Ω_f and $\psi^s|_{t=0} = \psi^{\text{in}}_s$ on $\partial\Omega_{fs}$.

Table of contents

1. Introduction
2. A new high-order framework for shallow-water asymptotics
 - Nonlinear shallow-water equations
 - Classical numerical methods for hyperbolic systems
 - High-order robust subcell monolithic DG/FV schemes
 - Arbitrary Lagrangian–Eulerian formulation
3. **Waves interacting with floating structures in shallow-water regimes**
 - Physical setting and constraints
 - Governing equations for water waves
 - Coupling with a floating object
 - Numerical scheme
4. Conclusion and perspectives



Slides available online at sachacardonna.github.io

Numerical challenges and solutions

What makes wave–structure simulations difficult?

We have to deal at the same time with **hyperbolic** NSW in Ω_f , **elliptic** PDE to compute DtN operator on $\partial\Omega_{fs}$ and **ODE systems**

↪ We need to design a high-order robust scheme for this **strongly coupled** system, which is pretty challenging!

Numerical challenges and solutions

What makes wave–structure simulations difficult?

We have to deal at the same time with **hyperbolic** NSW in Ω_f , **elliptic** PDE to compute DtN operator on $\partial\Omega_{fs}$ and **ODE systems**

↪ We need to design a high-order robust scheme for this **strongly coupled** system, which is pretty challenging!

Hyperbolic solver

We use the **subcell monolithic DG/FV** scheme previously developed.

Numerical challenges and solutions

What makes wave–structure simulations difficult?

We have to deal at the same time with **hyperbolic** NSW in Ω_f , **elliptic** PDE to compute DtN operator on $\partial\Omega_{fs}$ and **ODE systems**

↪ We need to design a high-order robust scheme for this **strongly coupled** system, which is pretty challenging!

Hyperbolic solver

We use the **subcell monolithic DG/FV** scheme previously developed.

Elliptic method

We rely on a **Hybrid High-Order** scheme for the DtN operator on $\partial\Omega_{fs}$

↪ **hybrid polynomial representation** and **static condensation**.

Numerical challenges and solutions

What makes wave–structure simulations difficult?

We have to deal at the same time with **hyperbolic** NSW in Ω_f , **elliptic** PDE to compute DtN operator on $\partial\Omega_{fs}$ and **ODE systems**

↪ We need to design a high-order robust scheme for this **strongly coupled** system, which is pretty challenging!

Hyperbolic solver

We use the **subcell monolithic DG/FV** scheme previously developed.

Elliptic method

We rely on a **Hybrid High-Order** scheme for the DtN operator on $\partial\Omega_{fs}$

↪ **hybrid polynomial representation** and **static condensation**.

Time-integration scheme

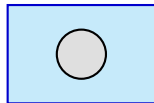
We integrate the coupled system in time using **SSP Runge-Kutta** method

↪ seen as convex combination of **forward Euler** steps.

A new wave–structure interaction numerical coupling

1. The following ODE is solved reading the values of elevation ζ and velocity $\mathbf{u}$ at the interface

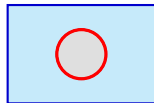
$$\partial_t \psi^s = -g\zeta - \frac{1}{2}|\mathbf{u}|^2 \quad \text{on } \partial\Omega_{fs} \quad \text{SSP-RK scheme}$$



A new wave–structure interaction numerical coupling

1. The following ODE is solved reading the values of elevation ζ and velocity $\mathbf{u}$ at the interface

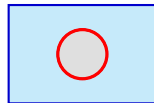
$$\partial_t \psi^s = -g\zeta - \frac{1}{2}|\mathbf{u}|^2 \quad \text{on } \partial\Omega_{fs} \quad \text{SSP-RK scheme}$$



A new wave–structure interaction numerical coupling

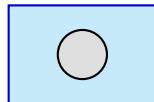
1. The following ODE is solved reading the values of elevation ζ and velocity $\mathbf{u}$ at the interface

$$\partial_t \psi^s = -g\zeta - \frac{1}{2}|\mathbf{u}|^2 \quad \text{on } \partial\Omega_{fs} \quad \text{SSP-RK scheme}$$



2. The trace ψ^s is used as a Dirichlet boundary condition to solve the interior elliptic problem

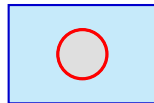
$$\begin{aligned} \nabla_{\mathbf{x}} \cdot (H^s \nabla_{\mathbf{x}} \phi^s) &= 0 & \text{in } \Omega_s, \\ \phi^s &= \psi^s & \text{on } \partial\Omega_{fs} \end{aligned} \quad \text{HHO scheme}$$



A new wave–structure interaction numerical coupling

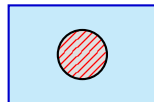
1. The following ODE is solved reading the values of elevation ζ and velocity $\mathbf{u}$ at the interface

$$\partial_t \psi^s = -g\zeta - \frac{1}{2}|\mathbf{u}|^2 \quad \text{on } \partial\Omega_{fs} \quad \text{SSP-RK scheme}$$



2. The trace ψ^s is used as a Dirichlet boundary condition to solve the interior elliptic problem

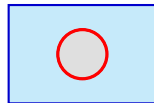
$$\begin{aligned} \nabla_{\mathbf{x}} \cdot (H^s \nabla_{\mathbf{x}} \phi^s) &= 0 & \text{in } \Omega_s, \\ \phi^s &= \psi^s & \text{on } \partial\Omega_{fs} \end{aligned} \quad \text{HHO scheme}$$



A new wave–structure interaction numerical coupling

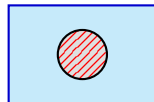
1. The following ODE is solved reading the values of elevation ζ and velocity $\mathbf{u}$ at the interface

$$\partial_t \psi^s = -g\zeta - \frac{1}{2}|\mathbf{u}|^2 \quad \text{on } \partial\Omega_{fs} \quad \text{SSP-RK scheme}$$



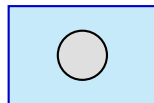
2. The trace ψ^s is used as a Dirichlet boundary condition to solve the interior elliptic problem

$$\begin{aligned} \nabla_{\mathbf{x}} \cdot (H^s \nabla_{\mathbf{x}} \phi^s) &= 0 & \text{in } \Omega_s, \\ \phi^s &= \psi^s & \text{on } \partial\Omega_{fs} \end{aligned} \quad \text{HHO scheme}$$



3. The normal flux at the interface is computed from the Dirichlet–Neumann operator $\Lambda\psi^s := H^s \nabla_{\mathbf{x}} \phi^s \cdot \mathbf{n}$ and used as a boundary condition for the exterior hyperbolic problem in Ω_f :

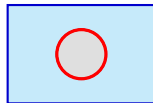
$$\begin{aligned} \partial_t \zeta^f + \nabla_{\mathbf{x}} \cdot (H^f \mathbf{u}^f) &= 0, \\ \partial_t \mathbf{u}^f + \nabla_{\mathbf{x}} \left(g\zeta^f + \frac{1}{2}|\mathbf{u}^f|^2 \right) &= \mathbf{0} \end{aligned} \quad \text{Monolithic scheme}$$



A new wave–structure interaction numerical coupling

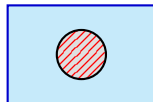
1. The following ODE is solved reading the values of elevation ζ and velocity $\mathbf{u}$ at the interface

$$\partial_t \psi^s = -g\zeta - \frac{1}{2}|\mathbf{u}|^2 \quad \text{on } \partial\Omega_{fs} \quad \text{SSP-RK scheme}$$



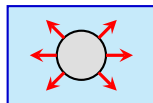
2. The trace ψ^s is used as a Dirichlet boundary condition to solve the interior elliptic problem

$$\begin{aligned} \nabla_{\mathbf{x}} \cdot (H^s \nabla_{\mathbf{x}} \phi^s) &= 0 & \text{in } \Omega_s, \\ \phi^s &= \psi^s & \text{on } \partial\Omega_{fs} \end{aligned} \quad \text{HHO scheme}$$



3. The normal flux at the interface is computed from the Dirichlet–Neumann operator $\Lambda\psi^s := H^s \nabla_{\mathbf{x}} \phi^s \cdot \mathbf{n}$ and used as a boundary condition for the exterior hyperbolic problem in Ω_f :

$$\begin{aligned} \partial_t \zeta^f + \nabla_{\mathbf{x}} \cdot (H^f \mathbf{u}^f) &= 0, \\ \partial_t \mathbf{u}^f + \nabla_{\mathbf{x}} \left(g\zeta^f + \frac{1}{2}|\mathbf{u}^f|^2 \right) &= \mathbf{0} \end{aligned} \quad \text{Monolithic scheme}$$



A collection of models and simulations

Model N°1 (current) → Fully-2D fixed pontoon & cylinder

In the model displayed here, the object **does not** move **vertically/horizontally**
↪ **see simulations N°5,6.**

A collection of models and simulations

Model N°1 (current) → Fully-2D fixed pontoon & cylinder

In the model displayed here, the object **does not** move **vertically/horizontally**
↪ **see simulations N°5,6.**

Model N°2 → Pseudo-1D heaving pontoon

The elliptic problem is formulated in terms of **Bernoulli pressure**, while the object **heaves** according to **Newton's second law**.
↪ **see simulations N°7–8.**

A collection of models and simulations

Model N°1 (current) → Fully-2D fixed pontoon & cylinder

In the model displayed here, the object **does not** move **vertically/horizontally**
↪ **see simulations N°5,6.**

Model N°2 → Pseudo-1D heaving pontoon

The elliptic problem is formulated in terms of **Bernoulli pressure**, while the object **heaves** according to **Newton's second law**.
↪ **see simulations N°7–8.**

Model N°3 → Fully-2D heaving cylinder

New!

Also in terms of **Bernoulli pressure** and **Newton's second law**, using a **Helmoltz-Hodge** decomposition → **see simulation N°9.**

A collection of models and simulations

Model N°1 (current) → Fully-2D fixed pontoon & cylinder

In the model displayed here, the object **does not** move **vertically/horizontally**
↪ **see simulations N°5,6.**

Model N°2 → Pseudo-1D heaving pontoon

The elliptic problem is formulated in terms of **Bernoulli pressure**, while the object **heaves** according to **Newton's second law**.
↪ **see simulations N°7–8.**

Model N°3 → Fully-2D heaving cylinder

New!

Also in terms of **Bernoulli pressure** and **Newton's second law**, using a **Helmutz-Hodge** decomposition → **see simulation N°9.**

Model N°4 → Pseudo-1D heaving–surging–pitching pontoon

Newest!

Also in terms of **Bernoulli pressure** and **Newton's second law**, using an integral 3×3 **added mass–inertia system** coupled with **ALE moving meshes** for $K \geq 1$ floating objects in order to simulate WEC farms.
↪ **see simulations N°10–12.**

Simulation of a WEC farm with $K = 5$ floating objects

- **Domain:** $\Omega = [-24, 184] \times [-20, 20]$ **Degree:** $k = 5$ **Mesh:** $n_{\text{el}} = 594$
- **Goal:** assessing the combined DG/FV-ALE-HHO heaving-surfing-pitching formulation on $K > 1$ floating objects

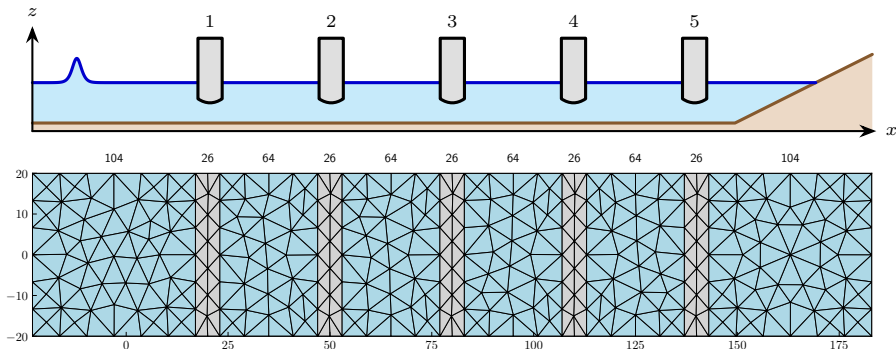


Figure: Schematic side view (top) and computational meshes (bottom) → **simulation N°12.**

Table of contents

1. Introduction
2. A new high-order framework for shallow-water asymptotics
 - Nonlinear shallow-water equations
 - Classical numerical methods for hyperbolic systems
 - High-order robust subcell monolithic DG/FV schemes
 - Arbitrary Lagrangian–Eulerian formulation
3. Waves interacting with floating structures in shallow-water regimes
 - Physical setting and constraints
 - Governing equations for water waves
 - Coupling with a floating object
 - Numerical scheme
4. Conclusion and perspectives



Slides available online at sachacardonna.github.io

Ongoing work and next steps

Work extending the thesis

- ▶ Studying the conservation properties of the **ALE scheme**,
- ▶ Analytical expression for the **DtN operator** in the cylinder case,
- ▶ Validating our coupled methods against **experimental data**,
- ▶ Studying the conversion of **wave-induced motion** into usable energy.

Ongoing work and next steps

Work extending the thesis

- ▶ Studying the conservation properties of the **ALE scheme**,
- ▶ Analytical expression for the **DtN operator** in the cylinder case,
- ▶ Validating our coupled methods against **experimental data**,
- ▶ Studying the conversion of **wave-induced motion** into usable energy.

Journal articles related to the thesis

- 📄 **S.C., A. Haidar, F. Marche & F. Vilar**, *Local subcell monolithic DG/FV methods for nonlinear SW models with source terms*. IJNMF, 2025.
- 📄 **S.C., F. Marche & F. Vilar**, *A high-order robust subcell monolithic DG/FV formulation for NSW equations on unstructured grids*. JCP (rev.), 2026.
- 📄 **S.C., F. Marche & F. Vilar**, *High-order well-balanced ALE numerical method for hyperbolic systems with source terms*. In preparation.

Ongoing work and next steps

Work extending the thesis

- ▶ Studying the conservation properties of the **ALE scheme**,
- ▶ Analytical expression for the **DtN operator** in the cylinder case,
- ▶ Validating our coupled methods against **experimental data**,
- ▶ Studying the conversion of **wave-induced motion** into usable energy.

Journal articles related to the thesis

- 📄 **S.C., A. Haidar, F. Marche & F. Vilar**, *Local subcell monolithic DG/FV methods for nonlinear SW models with source terms*. IJNMF, 2025.
- 📄 **S.C., F. Marche & F. Vilar**, *A high-order robust subcell monolithic DG/FV formulation for NSW equations on unstructured grids*. JCP (rev.), 2026.
- 📄 **S.C., F. Marche & F. Vilar**, *High-order well-balanced ALE numerical method for hyperbolic systems with source terms*. In preparation.
- 📄 **S.C., D. Lannes, & F. Marche**, *DtN operators for SW wave-structure interactions: modeling and DG-HHO approximation*. In preparation.
- 📄 **S.C., D. Lannes, F. Marche & F. Vilar**, *Combined subcell monolithic DG/FV-HHO-SSP-RK scheme for wave-structure interaction*. In preparation.

What are the plans for the future!

- Extension to **2D dispersive water-waves equations** (F. Marche),

What are the plans for the future!

- ▶ Extension to **2D dispersive water-waves equations** (F. Marche),
- ▶ **Surge-heave** model for the cylinder object (F. Marche, D. Lannes),

What are the plans for the future!

- ▶ Extension to **2D dispersive water-waves equations** (F. Marche),
- ▶ **Surge-heave** model for the cylinder object (F. Marche, D. Lannes),
- ▶ Monolithic frameworks for **(semi-)implicit** time-stepping (F. Vilar),

What are the plans for the future!

- ▶ Extension to **2D dispersive water-waves equations** (F. Marche),
- ▶ **Surge-heave** model for the cylinder object (F. Marche, D. Lannes),
- ▶ Monolithic frameworks for **(semi-)implicit** time-stepping (F. Vilar),
- ▶ **Godunov-Peshkov-Romenski** model for **wave-structure interaction**,

What are the plans for the future!

- ▶ Extension to **2D dispersive water-waves equations** (F. Marche),
- ▶ **Surge-heave** model for the cylinder object (F. Marche, D. Lannes),
- ▶ Monolithic frameworks for **(semi-)implicit** time-stepping (F. Vilar),
- ▶ **Godunov-Peshkov-Romenski** model for **wave-structure interaction**,
- ▶ **PINNs** and hybrid solvers for shallow-water flows (A. Haidar),

What are the plans for the future!

- ▶ Extension to **2D dispersive water-waves equations** (F. Marche),
- ▶ **Surge-heave** model for the cylinder object (F. Marche, D. Lannes),
- ▶ Monolithic frameworks for **(semi-)implicit** time-stepping (F. Vilar),
- ▶ **Godunov-Peshkov-Romenski** model for **wave-structure interaction**,
- ▶ **PINNs** and hybrid solvers for shallow-water flows (A. Haidar),
- ▶ Using **ALE mesh motion** to preserve involution (W. Boscheri),

What are the plans for the future!

- ▶ Extension to **2D dispersive water-waves equations** (F. Marche),
- ▶ **Surge-heave** model for the cylinder object (F. Marche, D. Lannes),
- ▶ Monolithic frameworks for **(semi-)implicit** time-stepping (F. Vilar),
- ▶ **Godunov-Peshkov-Romenski** model for **wave-structure interaction**,
- ▶ **PINNs** and hybrid solvers for shallow-water flows (A. Haidar),
- ▶ Using **ALE mesh motion** to preserve involution (W. Boscheri),
- ▶ **DG exterior calculus** based methods on manifolds (M. Hanot).

What are the plans for the future!

- ▶ Extension to **2D dispersive water-waves equations** (F. Marche),
- ▶ **Surge-heave** model for the cylinder object (F. Marche, D. Lannes),
- ▶ Monolithic frameworks for **(semi-)implicit** time-stepping (F. Vilar),
- ▶ **Godunov-Peshkov-Romenski** model for **wave-structure interaction**,
- ▶ **PINNs** and hybrid solvers for shallow-water flows (A. Haidar),
- ▶ Using **ALE mesh motion** to preserve involution (W. Boscheri),
- ▶ **DG exterior calculus** based methods on manifolds (M. Hanot).

Postdoc at University of Waterloo, Canada, with D. Del Rey Fernández
on **semigroup theory for PDEs** and **DDR methods**

~ Thank you for your attention! ~

Special acknowledgments

► Jury members ⚙️

↪ *D. Del Rey Fernández, J. Droniou, M. Dumbser, S. Gavrilyuk, J.-L. Guermond, M. Kazakova, B. Mohammadi & P. Noble*

► My mentors and advisors 🎓

↪ *Fabien & François*

► Supportive professors 👥

↪ *P. Azerad, W. Boscheri, J.-F. Crouzet, A. Duran, E. Gaburro, G. Gassner, D. Lannes, D. Le Roux, R. Loubère, V. Perrier...*

► Colleagues and friends 💬

↪ *Ali, Marien, Christina, Christelle, Orlane, Tanguy, Aurelio, Léo, Patrick, Joseph, Corentin, Bart', Amélie, Joris...*

✉ **Contact:** sacha.cardonna@umontpellier.fr
🌐 **Website:** sachacardonna.github.io



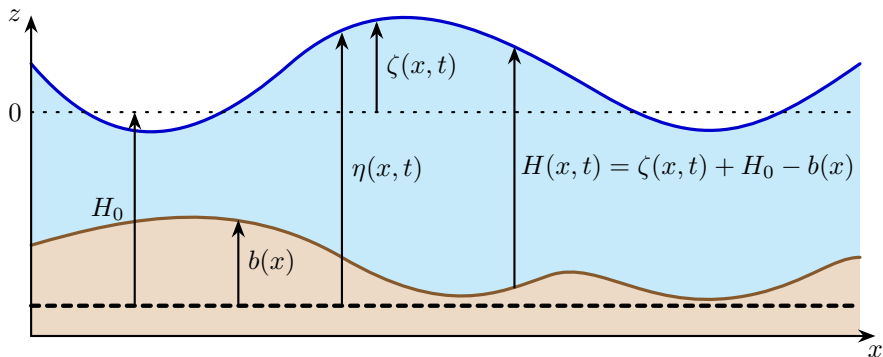


Figure: Shallow-water variables: rest depth H_0 , free surface elevation η , elevation relative to rest depth ζ , and total water depth H .

DG formulation through residuals

DG formulation

$$\psi_p^c \in \mathbb{P}^k(\omega_c)$$

$$\sum_{m=1}^{N_k} \frac{d\mathbf{v}_m^c}{dt} \int_{\omega_c} \psi_m^c \psi_p^c d\mathbf{x} - \int_{\omega_c} \mathbb{F}(\mathbf{v}_h^c) \cdot \nabla_{\mathbf{x}} \psi_p^c d\mathbf{x} + \int_{\partial\omega_c} \mathbb{F}^* \cdot \mathbf{n} \psi_p^c ds = \int_{\omega_c} \mathbf{B} \psi_p^c d\mathbf{x}$$

Residual DG formulation

$$\psi_m^c \in \mathbb{P}^k(\omega_c)$$

$$\mathbb{M}_c \frac{d\mathbf{V}_c}{dt} = \Xi_c + \mathbf{S}_c$$

$$\blacktriangleright (\mathbf{V}_c)_m = \mathbf{v}_m^c(t)$$

solution moments

$$\blacktriangleright (\mathbb{M}_c)_{mp} = \int_{\omega_c} \psi_m^c(\mathbf{x}) \psi_p^c(\mathbf{x}) d\mathbf{x}$$

local mass matrix

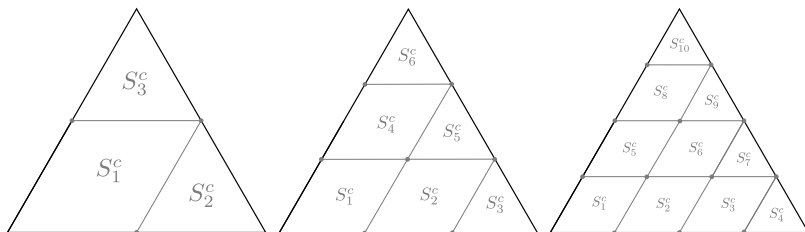
$$\blacktriangleright (\Xi_c)_m = \int_{\partial\omega_c} \mathbb{F}^* \cdot \mathbf{n} \psi_p^c ds - \int_{\omega_c} \mathbb{F}(\mathbf{v}_h^c) \cdot \nabla_{\mathbf{x}} \psi_p^c d\mathbf{x}$$

DG residuals

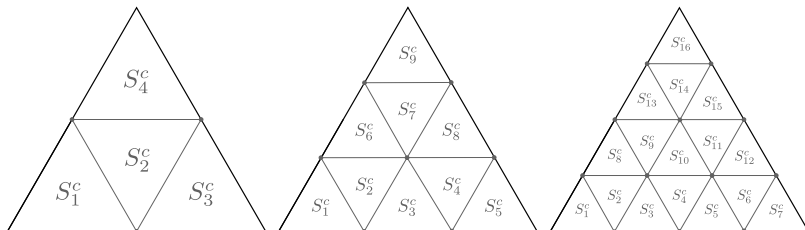
$$\blacktriangleright (\mathbf{S}_c)_m = \int_{\omega_c} \mathbf{B}(\mathbf{v}_h^c) \psi_p^c d\mathbf{x}$$

source term

Mesh subdivision



Cell ω_c subdivided into $N_s = N_k$ subcells for $\mathbb{P}^1$ (left), $\mathbb{P}^2$ (center) and $\mathbb{P}^3$ (right) cases



Cell ω_c subdivided into $N_s \geq N_k$ subcells for $\mathbb{P}^1$ (left), $\mathbb{P}^2$ (center) and $\mathbb{P}^3$ (right) cases

Subneighbors

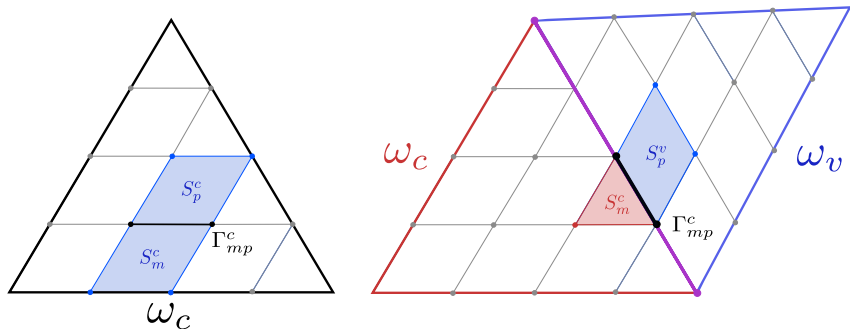


Figure: Two cases: subneighbor S_p inside cell ω_c (left), and subneighbor S_p inside neighbor cell ω_v (right).

Submean values and polynomial moments (1)

Mean value of a function over a subcell $S_m^c \subset \omega_c$

For any $f \in L^2(\omega_c)$, its submean value is $\bar{f}_m^c := \frac{1}{|S_m^c|} \int_{S_m^c} f(\mathbf{x}) \, d\mathbf{x}$.

Submean values and projection matrix

► $(\bar{\mathbf{V}}_c)_m = \bar{\mathbf{v}}_m^c(t)$ **submean values**

► $(\mathbb{P}_c)_{mp} = \frac{1}{|S_m^c|} \int_{S_m^c} \psi_p^c(\mathbf{x}) \, d\mathbf{x}$ **projection matrix**

$$\bar{\mathbf{v}}_m^c(t) = \frac{1}{|S_m^c|} \sum_{q=1}^{N_k} \mathbf{v}_q^c(t) \int_{S_m^c} \psi_q^c(\mathbf{x}) \, d\mathbf{x} \implies \boxed{\bar{\mathbf{V}}_c = \mathbb{P}_c \mathbf{V}_c}$$

⚠ $\mathbb{P}_c^\top \mathbb{P}_c$ has to be **non-singular**, so we use the least-square procedure:

$$\boxed{\mathbf{V}_c = \left(\mathbb{P}_c^\top \mathbb{P}_c \right)^{-1} \mathbb{P}_c^\top \bar{\mathbf{V}}_c}$$

If $N_s = N_k$, then $\bar{\mathbf{V}}_c = \mathbb{P}_c \mathbf{V}_c \Leftrightarrow \mathbf{V}_c = \mathbb{P}_c^{-1} \bar{\mathbf{V}}_c$.

Reconstructed DG fluxes (1)

Submean values vector derivative

$$\text{Since } \mathbf{M}_c \frac{d\mathbf{V}_c}{dt} = \Xi_c + \mathbf{S}_c \text{ and } \bar{\mathbf{V}}_c = \mathbf{P}_c \mathbf{V}_c \implies \boxed{\frac{d\bar{\mathbf{V}}_c}{dt} = \mathbf{P}_c \mathbf{M}_c^{-1} (\Xi_c + \mathbf{S}_c)}$$

Flux reconstruction to get a FV-like scheme

Let us consider the DG reconstructed flux $\hat{\mathbb{F}}_n$ such that

$$\begin{aligned} \frac{d\bar{\mathbf{v}}_m^c}{dt} &= -\frac{1}{|S_m^c|} \int_{\partial S_m^c} \hat{\mathbb{F}}_n(\mathbf{x}) \, d\mathbf{x} + (\mathbf{P}_c \mathbf{M}_c^{-1} \mathbf{S}_c)_m && \text{(FV-like scheme)} \\ &= -\frac{1}{|S_m^c|} \sum_{S_p^v \in \text{Neigh}(S_m^c)} \int_{\Gamma_{mp}^c} \hat{\mathbb{F}}_n(\mathbf{x}) \, d\mathbf{x} + (\mathbf{P}_c \mathbf{M}_c^{-1} \mathbf{S}_c)_m \\ &= -\frac{1}{|S_m^c|} \left(\sum_{S_p^v \in \check{\mathcal{V}}_m^c} \int_{\Gamma_{mp}^c} \hat{\mathbb{F}}_n(\mathbf{x}) \, d\mathbf{x} + \int_{\partial\omega_c \cap \partial S_m^c} \mathbb{F}_n^* \, d\mathbf{x} \right) + (\mathbf{P}_c \mathbf{M}_c^{-1} \mathbf{S}_c)_m \end{aligned}$$

under the hypothesis that $\hat{\mathbb{F}}_{n|\partial\omega} = \mathbb{F}^*$ for all $\omega \in \mathcal{T}_h$.

Reconstructed DG fluxes (2)

Interface reconstructed flux

We define $\widehat{\mathbb{F}}_{mp}$ at interface Γ_{mp}^c as:
$$\int_{\Gamma_{mp}^c} \widehat{\mathbb{F}}_n(\mathbf{x}) \, d\mathbf{x} = \varepsilon_{mp}^c \widehat{\mathbb{F}}_{mp},$$

where subface orientation is carried through ε_{mp}^c , such that $\varepsilon_{pm}^c = -\varepsilon_{mp}^c$.

Reconstructed flux system

$$-\mathbb{A}_c \widehat{\mathbb{F}}_c = \mathbb{D}_c \frac{d\overline{\mathbf{V}}_c}{dt} + \partial \mathbb{F}_c$$

$$\blacktriangleright (\widehat{\mathbb{F}}_c)_{mp} = |\Gamma_{mp}^c| \widehat{\mathbb{F}}_{mp}$$

interior subfaces fluxes

$$\blacktriangleright (\mathbb{A}_c)_{mp} = \varepsilon_{mp}^c$$

adjacency matrix

$$\blacktriangleright (\mathbb{D}_c)_m = |S_m^c|$$

subvolume matrix

$$\blacktriangleright (\partial \mathbb{F}_c)_m = \int_{\partial \omega_c \cap \partial S_m^c} \mathbb{F}_n^* \, d\mathbf{x}$$

cell boundary contribution

⚠ Since $\ker \mathbb{A}_c \neq \{\mathbf{0}\}$, we use a *Graph Laplacian technique*

Reconstructed DG fluxes (3)

Residual definition of reconstructed fluxes

$$\widehat{\mathbb{F}}_c = -\mathbb{A}_c^\top \mathcal{L}_c^{-1} (\mathbb{D}_c \mathbb{P}_c \mathbb{M}_c^{-1} \Xi_c + \partial \mathbb{F}_c)$$

where $\mathcal{L}_c^{-1}$ is the gen. inverse of $\mathbb{L}_c := \mathbb{A}_c \mathbb{A}_c^\top$ on the orthogonal of its kernel:

$$\mathcal{L}_c^{-1} = (\mathbb{L}_c + \lambda \Pi)^{-1} - \frac{1}{\lambda} \Pi, \quad \Pi = \frac{1}{N_s} (1 \otimes 1) \in \mathcal{M}_{N_k}, \quad \forall \lambda \neq 0$$

 **R. Abgrall**, *Some Remarks about Conservation for Residual Distribution Schemes*. Methods Appl. Math., 18:327-351, 2018.

Few remarks

- ▶ **Source term** is excluded in the definition since only flux-dependent integrals are considered in reconstruction,
- ▶ **Implementation**: only Ξ_c and boundary terms $\partial \mathbb{F}_c$ depend on time, but all the other terms are precomputable,
- ▶ **Alternative expression**: using spanning set of subresolution functions $\phi_m^c = p_{\omega_c}^k(1_m^c)$, where $p_{\omega_c}^k$ is the L^2 -projector on cell ω_c .

Remark about initialization strategy

Initialization strategy

Initialization is performed via **subcell averages** followed by projection using $\mathbb{P}_c$, instead of L^2 projection or interpolation as usually done in DG schemes

↪ this guarantees $\mathbf{v}_h \in \mathcal{H}_b^+$ at $t = 0$, and enforces $\eta_h = b_h$ in dry zones

⚠ Since b_h is discontinuous across cells, **hydrostatic reconstruction** is applied to both DG and subcell FV fluxes.

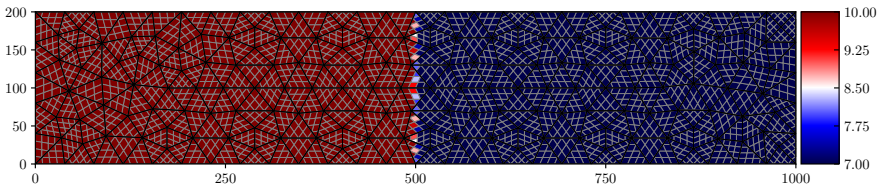


Figure: $\mathbb{P}^3$ dam-break problem initialization.

Two-scale hydrostatic reconstruction

Assuring both WB and positivity in numerical fluxes

❗ Hydrostatic reconstruction framework used on both DG and subcell FV fluxes
 ↪ ensures **positivity** of the water height, even for discontinuous topography

At each interface $\Gamma_{cv(k)}$ (resp. subinterface $\Gamma_{mp(k)}$), reconstructed values are defined:

- ▶ Topography rec.: $\tilde{b}_k = \max(b_k^-, b_k^+)$, $\check{b}_k = \tilde{b}_k - \max(0, \tilde{b}_k - \eta_k^-)$
- ▶ Water height/elevation rec.: $\check{H}_k^\pm = \max(0, \eta_k^\pm - \tilde{b}_k)$, $\check{\eta}_k^\pm = \check{H}_k^\pm + \tilde{b}_k$
- ▶ Modified states: $\check{\mathbf{v}}_k^\pm = \left(\check{\eta}_k^\pm, \frac{\check{H}_k^\pm}{H_k^\pm} \mathbf{q}_k^\pm \right)^\top$

These are then used in a Lax-Friedrichs-type flux $\mathbb{F}^*$, completed by a correction term $\check{\mathbb{F}}_{cv(k)}$ to ensure well-balancing:

$$\mathbb{F}_{cv(k)}^* = \mathbb{F}^*(\check{\mathbf{v}}_k^-, \check{\mathbf{v}}_k^+, \check{b}_k, \check{b}_k, \mathbf{n}_{cv(k)}) + \check{\mathbb{F}}_{cv(k)}$$

Source term treatment (1)

Flowchart of the discretization

💡 Dealing with both **polynomial DOFs** and **subcell-averaged values**

1. **Subcell averages**: compute the submean values of the quantities involved on each subcell, then reconstruct polynomials via projection matrix $\mathbb{P}_c$,
2. **Projection**: evaluate $B(v_h)$ at quadrature nodes, then apply an L^2 projection onto $\mathbb{P}^k$,
3. **Integration**: compute the mean value of the projected source over each subcell:

$$\overline{B}_m^c := \frac{1}{|S_m^c|} \int_{S_m^c} B_h \, dx$$

Implementation remark

Formally corresponds to multiplying the DG source integral by $\mathbb{P}_c \mathbb{M}_c^{-1}$:

$$\overline{B}_m^c = \mathbb{P}_c \mathbb{M}_c^{-1} \left(\int_{\omega_c} B_h \varphi_h \, dx \right)$$

Source term treatment (2)

Alternative discretization of the source term

$$\overline{B}_m^c = \overline{B}_m^{c,\text{FV}} + \Theta_m^c \left(\overline{B}_m^{c,\text{DG}} - \overline{B}_m^{c,\text{FV}} \right)$$

► $\Theta_m^c = \frac{1}{\#\text{Neigh}(S_m^c)} \sum_{S_p^v \in \text{Neigh}(S_m^c)} \Theta_{mp}$ **subcell global blending**

► $\overline{B}_m^{c,\text{DG}} = \frac{1}{|S_m^c|} \int_{S_m^c} B_h \, dx$ **DG source term**

► $\overline{B}_m^{c,\text{FV}} = \frac{1}{|S_m^c|} \int_{S_m^c} B(\overline{v}_m^c, \nabla_x b_h^c) \, dx$ **FV source term**

💬 No significant difference in results → we keep $\overline{B}_m^c = \overline{B}_m^{c,\text{DG}}$

Reformulation as a Godunov-like scheme

Solution at t^{n+1} as a convex combination of quantities defined at t^n

$$\begin{aligned}\bar{v}_m^{c,n+1} &= \bar{v}_m^{c,n} - \frac{\Delta t^n}{|S_m^c|} \sum_{S_p^v \in \text{Neigh}(S_m^c)} |\Gamma_{mp}| \tilde{\mathbb{F}}_{mp} + \Delta t^n \bar{B}_m^{c,n} \\ &+ \frac{\Delta t^n}{|S_m^c|} \mathbb{F}(\bar{v}_m^{c,n}) \cdot \sum_{S_p^v \in \text{Neigh}(S_m^c)} |\Gamma_{mp}| \mathbf{n}_{mp} \pm \frac{\sigma \Delta t^n}{|S_m^c|} \sum_{S_p^v \in \text{Neigh}(S_m^c)} |\Gamma_{mp}| \bar{v}_m^{c,n} \\ &= \left(1 - \frac{\sigma \Delta t^n}{|S_m^c|} \sum_{S_p^v \in \text{Neigh}(S_m^c)} |\Gamma_{mp}| \right) \bar{v}_m^{c,n} + \frac{\sigma \Delta t^n}{|S_m^c|} \sum_{S_p^v \in \text{Neigh}(S_m^c)} |\Gamma_{mp}| \tilde{v}_{mp}^{*, -} + \Delta t^n \bar{B}_m^{c,n}\end{aligned}$$

► $\tilde{v}_{mp}^*$ are the **blended Riemann intermediate states**

$$\tilde{v}_{mp}^* := \bar{v}_m^{c,n} - \frac{\tilde{\mathbb{F}}_{mp} - \mathbb{F}(\bar{v}_m^{c,n}) \cdot \mathbf{n}_{mp}}{\sigma} = v_{mp}^* - \Theta_{mp} \left(\frac{\hat{\mathbb{F}}_{mp} - \mathbb{F}^{*, \text{FV}}_{mp}}{\sigma} \right),$$

► v_{mp}^* are the 1st-order **FV Riemann intermediate states**.

Analytical formula to ensure water height positivity

Relying on 1st-order FV Riemann intermediate states

Proof of the natural **preservation of water-height positivity** for 1st-order elevation Riemann FV states $\eta_{mp}^{*,\pm}$

↪ allows us to rely on the **robustness of FV framework** to ensure the properties we want

Physical admissibility detector

$$\Theta_{mp}^{\mathcal{H}_b^+} := \min \left(\Theta_{mp}^{\mathcal{H}_b^+, -}, \Theta_{mp}^{\mathcal{H}_b^+, +} \right)$$

$$\blacktriangleright \Theta_{mp}^{\mathcal{H}_b^+, -} := \frac{\sigma \left(\eta_{mp}^* - \bar{b}_m^c \right)}{\Delta F_{mp}} \quad \text{if } \Delta F_{mp} > 0, \quad \Theta_{mp}^{\mathcal{H}_b^+, -} = 1 \quad \text{else;}$$

$$\blacktriangleright \Theta_{mp}^{\mathcal{H}_b^+, +} := \frac{\sigma \left(\bar{b}_p^v - \eta_{mp}^* \right)}{\Delta F_{pm}} \quad \text{if } \Delta F_{pm} < 0, \quad \Theta_{mp}^{\mathcal{H}_b^+, +} = 1 \quad \text{else.}$$

Analytical formula to prevent spurious oscillations

Mimicking a local maximum principle

$$\alpha_m^c := \min_{S_p^v \in \mathcal{N}(S_m^c)} \bar{\eta}_p^{v,n} \leq \bar{\eta}_m^{c,n+1} \leq \max_{S_p^v \in \mathcal{N}(S_m^c)} \bar{\eta}_p^{v,n} =: \beta_m^c$$

where $\mathcal{P}_m^c$ is the set of vertices $\mathbf{x}_p$ of subcell S_m^c and

$$\mathcal{N}(S_m^c) := \bigcup_{\mathbf{x}_p \in \mathcal{P}_m^c} \{S_q \mid \mathbf{x}_p \in S_q\}$$

Subcell numerical admissibility detector

$$\Theta_{mp}^{\text{SubNAD}} := \min \left(1, \left| \frac{\sigma}{\Delta F_{mp}} \right| \begin{cases} \min(\beta_p^v - \eta_{mp}^{*,+}, \eta_{mp}^{*, -} - \alpha_m^c) & \text{if } \Delta F_{mp} > 0 \\ \min(\beta_m^c - \eta_{mp}^{*, -}, \eta_{mp}^{*, +} - \alpha_p^v) & \text{if } \Delta F_{mp} < 0 \end{cases} \right)$$

⚠ For NSW, no local maximum principle for the conserved variable!

↪ needs to be **relaxed** in the presence of **smooth extremas**

Blending relaxation

Smoothness detector on subcells

⚠ Must ensure to **relax correction** close to **smooth extremas**!

1. **Linearized reconstructions** of $\partial_x \eta_h$ and $\partial_y \eta_h$ are built inside subcell, i.e.

$$\mathfrak{E}_x^m(\mathbf{x}) := \overline{\partial_x \eta_h^c}^m + \overline{\nabla_{\mathbf{x}}(\partial_x \eta_h^c)}^m \cdot (\mathbf{x} - \mathbf{x}_m^c),$$

$$\mathfrak{E}_y^m(\mathbf{x}) := \overline{\partial_y \eta_h^c}^m + \overline{\nabla_{\mathbf{x}}(\partial_y \eta_h^c)}^m \cdot (\mathbf{x} - \mathbf{x}_m^c),$$

2. **Checking these indicators values at vertices**: if their values remain within the local extrema over a stencil, the solution is considered smooth over subcell S_m^c ,
3. **If two neighbors S_m^c and S_p^v are considered smooth**, the blending coefficient on Γ_{mp} is relaxed to $\Theta_{mp} = 1 \rightarrow$ **full DG accuracy**

When to use the smooth extrema detector?

- ▶ **Order ≤ 2** : No detector needed.
- ▶ **Order = 3**: Detector required at the **cell** level.
- ▶ **Order ≥ 4** : Detector required at the **subcell** level.

Remark about blending smoothening

Why smoothening blending coefficient?

A **sharp switch** between low and high-order fluxes may cause **local oscillations**

↪ blending smoothers designed to **mitigate abrupt transitions**

► **Mean-value smoother** (default in experiments):

$$\Theta_m^c = \frac{1}{\#\text{Neigh}(S_m^c)} \sum_{S_p^v \in \text{Neigh}(S_m^c)} \Theta_{mp},$$

$$\tilde{\Theta}_{mp} = \min \left(\Theta_{mp}, \frac{1}{\#\mathcal{V}_{mp}} \sum_{S_q^v \in \mathcal{V}_{mp}} \Theta_q^v \right)$$

↪ Less diffusive, smoother transitions

► **Minimum-value smoother:**

$$\Theta_m^c = \min_{S_p^v \in \text{Neigh}(S_m^c)} \Theta_{mp},$$

$$\tilde{\Theta}_{mp} = \min \left(\Theta_{mp}, \min_{S_q^v \in \text{Neigh}(S_m^c \cup S_p^v)} \Theta_q^v \right)$$

↪ Stronger damping near discontinuities

Preservation of steady-states (1)

Why does it matter ?

- ▶ **Preserves lake at rest steady states exactly**, avoiding spurious motions;
- ▶ **Reduces numerical errors** near equilibrium, especially when small perturbations are present;
- ▶ **Essential for wet/dry interfaces**, where small oscillations can destabilize the scheme.

Well-balancing (WB) property

Providing that the integrals of discrete formulation are exactly computed, we have the following result:

$$\forall n \in \mathbb{N}, \quad \forall \eta^e > 0, \quad (\eta_h^n = \eta^e \text{ and } \mathbf{q}_h^n = \mathbf{0}) \implies (\eta_h^{n+1} = \eta^e \text{ and } \mathbf{q}_h^{n+1} = \mathbf{0})$$

Preservation of steady-states (2)

Sketch of proof

Objective: showing that numerical fluxes are cancelling the source term *i.e.*

$$\frac{1}{|S_m^c|} \sum_{S_p^v \in \text{Neigh}(S_m^c)} |\Gamma_{mp}| \tilde{\mathbb{F}}_{mp} = \overline{\mathbf{B}}_m^{c,n} \quad \text{s.t.} \quad \overline{\mathbf{v}}_m^{c,n+1} = \overline{\mathbf{v}}_m^{c,n}.$$

- ▶ Exact integration required → natural with high-order quadrature;
- ▶ Under well-balanced assumptions:

$$\nabla_{\mathbf{x}} \cdot \mathbb{F}(\mathbf{v}_c, b_c) = \mathbf{B}[b_c](\mathbf{v}_c), \quad \forall \omega_c \subset \Omega_f,$$

- ▶ Fluxes $\hat{\mathbb{F}}_{mp}$ and $\mathbb{F}_{mp}^{*,FV}$ match the continuous flux $\mathbb{F}_h^c \cdot \mathbf{n}_{mp}$ under equilibrium;
- ▶ $\tilde{\mathbb{F}}_{mp}$ is built as a convex combination of these well-balanced fluxes
 $\hookrightarrow$ preserves equilibrium as well !

Order of accuracy assessment

Steady vortex with $\mathcal{C}^\infty$ topography

- **Domain:** $\Omega = [-5, 5]^2$ **Degree:** $k = 1, 2, 3$ **Mesh:** $n_{\text{el}} = 200 \rightarrow 12800$
- **Goal:** convergence of the scheme on a smooth solution with a consistent discretization of the topography source term

k	1		2		3	
h	$E_{L^2}^\eta$	$q_{L^2}^\eta$	$E_{L^2}^\eta$	$q_{L^2}^\eta$	$E_{L^2}^\eta$	$q_{L^2}^\eta$
1	9.445E-2	2.35	1.529E-2	2.91	4.580E-3	4.19
$\frac{1}{2}$	1.854E-2	2.16	2.039E-3	3.03	2.505E-4	4.10
$\frac{1}{4}$	4.158E-3	2.07	2.491E-4	2.97	1.465E-5	4.00
$\frac{1}{8}$	9.923E-4	—	3.187E-5	—	9.165E-7	—

Figure: L^2 -errors between numerical and analytical solutions and convergence rates for η at time $t = 0.1$ sec.

Riemann problem (dam-break on a wet bed)

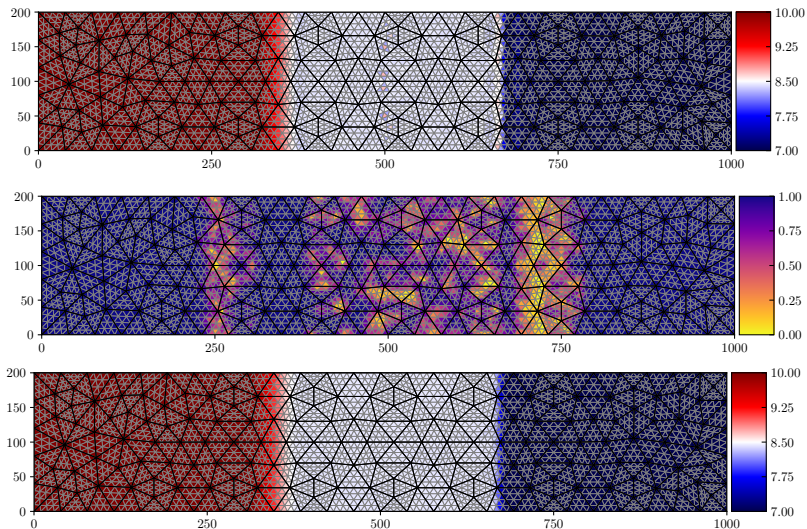


Figure: At $t = 18$ sec, $\mathbb{P}^4$ unlimited DG elevation (top), map of blending coefficient means per subcell (center) and monolithic DG/FV subcells elevation (bottom).

Hybrid High-Order (HHO) scheme for the elliptic problems

Elliptic solver for the Dirichlet–Neumann operator

In the general geometry, the Dirichlet–Neumann operator is not explicit; we therefore compute it by solving the interior problem

$$\nabla_{\mathbf{x}} \cdot (H^s \nabla_{\mathbf{x}} \phi^s) = 0 \quad \text{in } \Omega_s, \quad \phi^s = \psi^s \quad \text{on } \partial\Omega_{fs}.$$

↪ **Hybrid High-Order** (HHO) method is used to solve this elliptic equation.

Discrete HHO unknowns

Cell and face unknowns (polynomials of degree k):

$$\phi_c^s \in \mathbb{P}^k(\omega_c), \quad \phi_\Gamma^s \in \mathbb{P}^k(\Gamma), \quad \omega_c^s \subset \Omega_s, \quad \Gamma \subset \partial\omega_c^s.$$

- ▶ **Hybrid viewpoint:** cell unknowns represent the interior behaviour, while face unknowns control how neighbouring cells communicate,
- ▶ **Local reconstruction:** a high-order approximation of the gradient is built inside each cell, reproducing the continuous integration-by-parts structure,
- ▶ **Geometric flexibility:** naturally supports arbitrary polyhedral meshes and non-matching interfaces, simplifying mesh generation.

Structure of the discrete HHO problem

Global formulation

We introduce the **global HHO space of degree k** :

$$\mathbf{U}_h^{s,k} := \left\{ \underline{\phi}_h^s = ((\phi_c^s)_c, (\phi_\Gamma^s)_\Gamma) \mid \phi_c^s \in \mathbb{P}^k(\omega_c^s), \phi_\Gamma^s \in \mathbb{P}^k(\Gamma) \right\}.$$

The discrete interior problem reads: find $\underline{\phi}_h^s \in \mathbf{U}_{h,\text{Dir}}^{s,k}$ such that

$$\mathcal{A}_h^s(\underline{\phi}_h^s, \underline{w}_h) = \sum_{\omega_c^s \subset \Omega_s} \mathcal{A}_c^s(\phi_c^s, \underline{w}_c) = 0, \quad \forall \underline{w}_h \in \mathbf{U}_{h,0}^{s,k}.$$

Algebraic formulation

The bilinear form $\mathcal{A}_h^s$ translates into stiffness blocks $\mathbb{K}_{\alpha\beta}$ (gradient reconstructions + stabilization), while Ψ^s contains the Dirichlet boundary data ψ^s :

$$\begin{bmatrix} \mathbb{K}_{cc} & \mathbb{K}_{c\Gamma} \\ \mathbb{K}_{\Gamma c} & \mathbb{K}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \underline{\phi}_c^s \\ \phi_\Gamma^s \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \Psi^s \end{bmatrix}$$

↪ **Static condensation**: since $\mathbb{K}_{cc}$ is **block-diagonal** (cells only interact via faces), cell unknowns $\underline{\phi}_c^s$ are locally eliminated to solve a global system on faces only:

$$(\mathbb{K}_{\Gamma\Gamma} - \mathbb{K}_{\Gamma c} \mathbb{K}_{cc}^{-1} \mathbb{K}_{c\Gamma}) \phi_\Gamma^s = \Psi^s$$

Discrete DtN operator for the pontoon case

Discrete Fourier transform for efficient evaluation

When the solution **genuinely depends** on y , the explicit operator can be evaluated efficiently by **Fast Fourier Transform (FFT)**, keeping the pontoon case computationally inexpensive.

Transverse-invariant case: reduction to a 1D model

Assume that the solution **does not** depend on y , then the fully two-dimensional pontoon model reduces to a classical **one-dimensional wave-structure interaction system**, where the interior motion is described by a single discharge variable

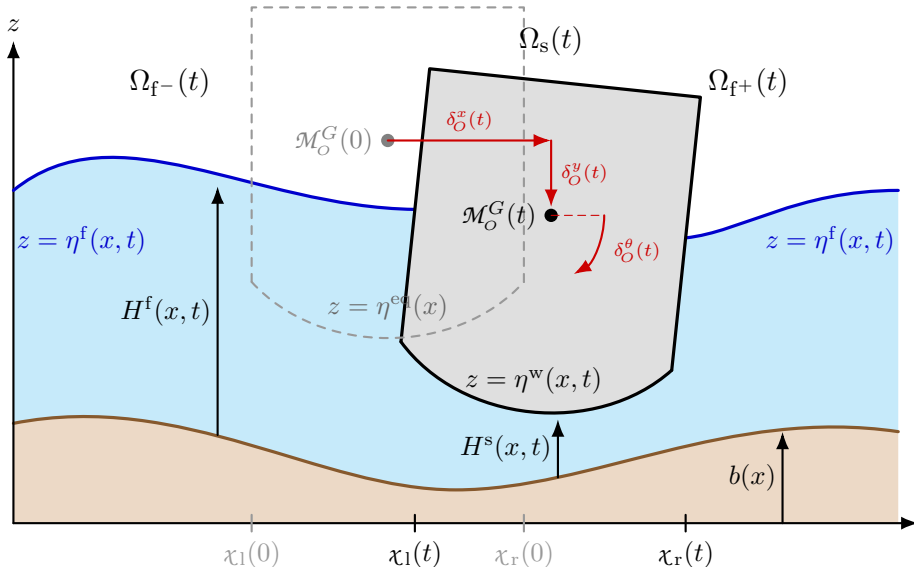
$$q^s(t) = \frac{H^s}{2l} (\psi_+ - \psi_-),$$

which satisfies

$$\frac{dq^s}{dt}(t) = -\frac{H^s}{2l} (\Pi_+ - \Pi_-).$$

↪ The structure behaves simply as a **dynamic transmission law** between the left and right waves!

Heaving–surging–pitching pontoon model (1)



Heaving–surging–pitching pontoon model (2)

Rigid-body quantities

For a point of the wetted surface, $\mathbf{r}_G := (r_x, r_z)^\top$ denotes its position relative to the center of mass and $\mathbf{T}_G := (r_z, -r_x, -\frac{1}{2}|\mathbf{r}_G|^2)^\top$ the associated rigid-body transport vector. The **dry mass–inertia matrix** is

$$\mathbb{M}_0 := \text{diag}(m_o, m_o, i_o),$$

where m_o and i_o denote the mass and moment of inertia of the pontoon.

Hydrodynamic added mass and inertia

For any $\phi \in L^1(\Omega_s(t); \mathbb{R}^3)$, we define $\langle \phi \rangle_{H^{-1}} := (\int_{\Omega_s(t)} \frac{\phi}{H^w} d\mathbf{x})(\int_{\Omega_s(t)} \frac{d\mathbf{x}}{H^w})^{-1}$ and $\mathbf{T}_G^* := \mathbf{T}_G - \langle \mathbf{T}_G \rangle_{H^{-1}}$. The integral added mass–inertia matrix is then

$$\mathbb{M}_a(t) := \rho \int_{\Omega_s(t)} \frac{\mathbf{T}_G^* \otimes \mathbf{T}_G^*}{H^w} d\mathbf{x}$$

Here, $\Omega_s(t)$ is the wetted interior domain, H^w the water height below the pontoon and ρ the fluid density.

Coupled rigid-body dynamics

Three-degree-of-freedom Newton system

The three rigid-body accelerations are determined simultaneously through

$$(\mathbb{M}_0 + \mathbb{M}_a(t)) \begin{pmatrix} \ddot{\delta}_O^x \\ \ddot{\delta}_O^y \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} \mathcal{F}_{\text{hyd}}^x \\ \mathcal{F}_{\text{hyd}}^y - m_O g \\ \mathcal{M}_{\text{hyd}}^G \end{pmatrix} \quad \begin{matrix} \text{surge,} \\ \text{heave,} \\ \text{pitch.} \end{matrix}$$

The terms $\mathcal{F}_{\text{hyd}}^x$, $\mathcal{F}_{\text{hyd}}^y$ and $\mathcal{M}_{\text{hyd}}^G$ are the horizontal force, vertical force and moment reconstructed from the interior **HHO pressure** and the lateral pressure resultants.

Recovery of the Chapter 5 model

For $\delta_O^x = \theta = 0$, the vertical component reduces exactly to

$$\tau_{\text{buoy}}^2 \ddot{\delta}_O^y + \delta_O^y = \frac{1}{2\rho g l L_y} \int_{\Omega_s} \Pi^s \, dx \quad \tau_{\text{buoy}}^2 := \frac{m_O}{2\rho g l L_y}.$$

Thus, the heaving equation of Chapter 5 is recovered, with Π^s the interior Bernoulli pressure, $2l$ the pontoon width and L_y its transverse extension.